HSS-Iteration-Based Iterative Interpolation of Curves and Surfaces with NTP Bases

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Abstract. Based on the Hermitian and skew-Hermitian splitting (HSS) iteration technique [14], a new iterative interpolation technique called HPIA for curves and surfaces with NTP bases and its weighted version WHPIA are proposed. We take the previous iteration and the current iteration into account simultaneously, and establish a function based on NTP bases as a perturbation term in the iteration process. Convergence analyses and the approximate optimal weight of WHPIA are given. Theoretical and experimental results show that HPIA and WHPIA are effective.

Keywords: HSS iteration · NTP bases · PIA · Interpolation · Convergence

1 Introduction

Essentially, as a popular data fitting technique in recent years, geometric iteration is an iterative method for solving linear equations in linear algebra. Since the geometric iteration method was proposed, it has been widely used in the academic research and engineering practices in the geometric design and related fields [1–5]. By using the technique of geometric iteration, not only achieved better results by addressing traditional problems of geometric design, such as offset curves, degree reduction, and polynomial approximation to rational curves and surfaces and etc., but also has been successfully applied to adaptive data fitting, large scale data fitting, symmetric surface fitting, generation of curves interpolating given positions, tangent, and curvature vectors, generation of quality guaranteed quadrilateral and hexahedral meshes, generation of trivariate B-spline solids.

The technique of geometric iteration in geometric design was originated and developed by Lin et al. [6–13]. In 2004, Lin proved the property of profit-and-loss for non-uniform cubic B-spline curves and surfaces [6], and for blending curves and tensor product blending patches with normalized totally positive(NTP)

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bases in 2005 [7]. The approach of geometric iteration is called progressive iterative approximation(PIA) in [7], which addresses both interpolation and approximation (including EPIA [8] and LSPIA [5,9]). Lin has given the PIA iterative formats with NTP bases in [6,7] as follows,
the case of curve:

$$\begin{aligned}&[\boldsymbol{A}_2^{k+1}, \boldsymbol{A}_3^{k+1}, \cdots, \boldsymbol{A}_{n-1}^{k+1}]\\=&(\boldsymbol{I}-\boldsymbol{N})[\boldsymbol{A}_2^{k}, \boldsymbol{A}_3^{k}, \cdots, \boldsymbol{A}_{n-1}^{k}], k=0,1,\cdots\end{aligned} \tag{1}$$

the case of surface:

$$\begin{aligned}&[\boldsymbol{A}_{11}^{k+1}, \boldsymbol{A}_{12}^{k+1}, \cdots, \boldsymbol{A}_{1n}^{k+1}, \cdots, \boldsymbol{A}_{m1}^{k+1}, \boldsymbol{A}_{m2}^{k+1}, \cdots, \boldsymbol{A}_{mn}^{k+1}]\\=&(\boldsymbol{I}-\boldsymbol{N})[\boldsymbol{A}_{11}^{k}, \boldsymbol{A}_{12}^{k}, \cdots, \boldsymbol{A}_{1n}^{k}, \cdots, \boldsymbol{A}_{m1}^{k}, \boldsymbol{A}_{m2}^{k}, \cdots, \boldsymbol{A}_{mn}^{k}]\\&, k=0,1,\cdots, \boldsymbol{N}=\boldsymbol{N}_1 \otimes \boldsymbol{N}_2\end{aligned} \tag{2}$$

where the difference vector $\boldsymbol{A}_i^k$ or $\boldsymbol{A}_{ij}^k$ is calculated as

$$\begin{cases}\boldsymbol{A}_i^k = (\boldsymbol{Q}_i - \boldsymbol{C}^k(u_i))\\ \boldsymbol{A}_{ij}^k = (\boldsymbol{Q}_{ij} - \boldsymbol{C}^k(u_i, v_j))\end{cases}$$

On the other hand, (1) and (2) can be written in matrix form as $\boldsymbol{P}^k = (\boldsymbol{I} - \boldsymbol{N})\boldsymbol{P}^{k-1} + \boldsymbol{Q}$, where $\boldsymbol{P}^k$ is a column vector of control vertexes, $\boldsymbol{I}$ is identity matrix, $\boldsymbol{N}$ is totally positive(TP) collocation matrix and $\boldsymbol{Q}$ is a column vector of data points. In [10], Lu present a new and efficient method for weighted PIA of data points by using NTP bases. The progress can be written in matrix form as $\boldsymbol{P}^k = (\boldsymbol{I} - \omega\boldsymbol{N})\boldsymbol{P}^{k-1} + \boldsymbol{Q}$. And he proved that the weighted PIA based on an NTP basis of the space has the fastest convergence rate when

$$\omega = \frac{2}{1 + \lambda_n(\boldsymbol{N})},$$

where $\lambda_n(\boldsymbol{N})$ is the smallest eigenvalue of $\boldsymbol{N}$.

Bai et al. proposed the use of the Hermitian/skew-Hermitian splitting(HSS) iteration method [14]. Theoretical analysis has shown that this HSS-iteration converges unconditionally to the exact solution of the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Based on the HSS-iteration technique, we present a new iteration method and its weighted version for progressive iteration approximation of data points by using NTP bases and prove their convergence. The iterative process of these two methods consists of two steps, and the iterative difference vectors in the two steps are different from each other. For convenience, we call them HPIA and WHPIA, respectively.

2 Iterative Format of HPIA

2.1 The Case of Curves

Given an NTP basis $\{\boldsymbol{N}_i(u)\}_{i=0}^n$ and a control vertexes set $\{\boldsymbol{P}_i^0\}_{i=0}^n$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, we can generate the initial curve

$$\boldsymbol{C}^0(u) = \sum_{i=0}^{n} \boldsymbol{P}_i^0 N_i(u),$$

We assign control vertexes set $\{\boldsymbol{P}_i^0\}_{i=0}^n$ with a real increasing parameters set $\{u_i\}_{i=0}^n$, i.e. $u_0 < u_1 < \cdots < u_n$.

Then, the remaining curves of the sequence, $\boldsymbol{C}^k(u)$ for $k \geq 1$, can be calculated as follows

$$\boldsymbol{C}^k(u) = \sum_{i=0}^{k} \boldsymbol{P}_i^k N_i(u),$$

where

$$\begin{cases} \boldsymbol{P}_i^k = \boldsymbol{P}_i^{k-1/2} + \boldsymbol{\Delta}_{1i}^k \\ \boldsymbol{\Delta}_{1i}^k = \boldsymbol{P}_i^0 - [\frac{1}{2}(\boldsymbol{C}^k(u_i) + \boldsymbol{C}^{k-1/2}(u_i)) \\ \qquad + \frac{1}{2}(\boldsymbol{C}^{k-1/2}(N_i) - \boldsymbol{C}^k(N_i))] \\ \boldsymbol{P}_i^{k-1/2} = \boldsymbol{P}_i^{k-1} + \boldsymbol{\Delta}_{2i}^k \\ \boldsymbol{\Delta}_{2i}^k = \boldsymbol{P}_i^0 - [\frac{1}{2}(\boldsymbol{C}^{k-1}(u_i) + \boldsymbol{C}^{k-1/2}(u_i)) \\ \qquad + \frac{1}{2}(\boldsymbol{C}^{k-1/2}(N_i) - \boldsymbol{C}^{k-1}(N_i))] \end{cases} \tag{3}$$

In (3), $\boldsymbol{C}^k(N_i)$ is defined as follows

$$\boldsymbol{C}^k(N_i) = \sum_{j=0}^{n} \boldsymbol{P}_j^k N_i(u_j), i = 0, \cdots, n, k = 0, \frac{1}{2}, 1, \cdots,$$

which is a function that takes bases as variables. Since $\lim_{k\to\infty} \{\boldsymbol{P}_i^{k-1/2}\}_{i=0}^n = \lim_{k\to\infty} \{\boldsymbol{P}_i^k\}_{i=0}^n$, we have $\lim_{k\to\infty} (\boldsymbol{C}^{k-1/2}(N_i) - \boldsymbol{C}^k(N_i)) = 0$. Here, we consider $\boldsymbol{C}^k(N_i)$ as a perturbation term in the iteration process.

We call (3) as HPIA format of curves, which consists of two steps and replaces the iterative step length of each control vertex in PIA.

2.2 The Case of Surfaces

Given two NTP bases $\{N_i(u)\}_{i=0}^n$, $\{S_j(v)\}_{j=0}^m$ and a control vertexes set $\{\boldsymbol{P}_{ij}^0\}_{i=0,j=0}^{n,m}$ in $\mathbb{R}^3$, we can generate the initial surface

$$\boldsymbol{C}^0(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \boldsymbol{P}_{ij}^0 N_i(u) S_j(v).$$

We assign control vertexes set $\{\boldsymbol{P}_{ij}^0\}_{i=0,j=0}^{n,m}$ with two real increasing parameters set $\{u_i\}_{i=0}^{n}$,i.e., $u_0 < u_1 < \cdots < u_n$ and $\{v_j\}_{j=0}^{m}$,i.e., $v_0 < v_1 < \cdots < v_m$.

Like the case of curves, we can take $\boldsymbol{C}^k(N,S) = \sum_{i=0}^{n}\sum_{j=0}^{m}\boldsymbol{P}_{ij}^k N(u_i) S(v_j), k = 0,1,\cdots$ as a perturbation term. Then, the remaining surfaces of the sequence, $\boldsymbol{C}^k(u,v)$ for $k \geq 1$, can be calculated as follows

$$\boldsymbol{C}^k(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m}\boldsymbol{P}_{ij}^k N_i(u) S_j(v).$$

where

$$\begin{cases} \boldsymbol{P}_{ij}^k = \boldsymbol{P}_{ij}^{k-1/2} + \boldsymbol{\Delta}_{1ij}^k \\ \boldsymbol{\Delta}_{1ij}^k = \boldsymbol{P}_{ij}^0 - [\frac{1}{2}(\boldsymbol{C}^k(u_i,v_j) + \boldsymbol{C}^{k-1/2}(u_i,v_j)) \\ \qquad + \frac{1}{2}(\boldsymbol{C}^{k-1/2}(N_i,S_j) - \boldsymbol{C}^k(N_i,S_j))] \\ \boldsymbol{P}_{ij}^{k-1/2} = \boldsymbol{P}_{ij}^{k-1} + \boldsymbol{\Delta}_{2ij}^k \\ \boldsymbol{\Delta}_{2ij}^k = \boldsymbol{P}_{ij}^0 - [\frac{1}{2}(\boldsymbol{C}^{k-1}(u_i,v_j) + \boldsymbol{C}^{k-1/2}(u_i,v_j)) \\ \qquad + \frac{1}{2}(\boldsymbol{C}^{k-1/2}(N_i,S_j) - \boldsymbol{C}^{k-1}(N_i,S_j))] \end{cases} \tag{4}$$

We call (4) as HPIA format of surfaces, which, like the case of curves, also consists of two steps and replaces the iterative step length of each control vertex in PIA.

Remark 1. From (3) and (4), we will get $\lim_{k\to\infty}\boldsymbol{C}^k(u_i) = \boldsymbol{P}_i^0$ or $\lim_{k\to\infty}\boldsymbol{C}^k(u_i,v_j) = \boldsymbol{P}_{ij}^0$, If for any $\varepsilon > 0$, there is a natural number T, when k, $s > T$,$\|\boldsymbol{P}_i^k - \boldsymbol{P}_i^{k-1}\| < \varepsilon$ or $\|\boldsymbol{P}_{ij}^k - \boldsymbol{P}_{ij}^{k-1}\| < \varepsilon$.

3 Convergence Analysis

Lemma 1. *Given any two non-singular collocation matrices* $\boldsymbol{N}_1 = (N_j(u_i))_{i,j=0}^{n,n}$, $\boldsymbol{N}_2 = (S_j(v_i))_{i,j=0}^{m,m}$, *which is defined on two NTP bases* $\{N_j(u)\}_{j=0}^{n}$, $\{S_j(v)\}_{j=0}^{m}$. *And assuming that* $\lambda_i(\boldsymbol{N}_1), i = 0,1,\cdots,n, \lambda_i(\boldsymbol{N}_2), i = 0,1,\cdots,m$ *are their eigenvalues respectively. Then,*

(1) $0 < \lambda_i(\boldsymbol{N}_1) \leq 1, 0 < \lambda_i(\boldsymbol{N}_2) \leq 1,$
(2) $0 < \lambda_i(\boldsymbol{N}_1 \otimes \boldsymbol{N}_2) \leq 1,$ *here* $\otimes$ *is Kronecker product.*

The proof of this Lemma 1 can be found in Theorems 2.1 and 2.2 of [2].
The two iterative processes of (3) and (4) can be written in matrix form

$$\begin{cases} \left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_-\right)\boldsymbol{P}^k = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\boldsymbol{P}^{k-1/2} + \boldsymbol{P}^0 \\ \left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)\boldsymbol{P}^{k-1/2} = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_-\right)\boldsymbol{P}^{k-1} + \boldsymbol{P}^0 \end{cases} \tag{5}$$

where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{N}_+ = \boldsymbol{N} + \boldsymbol{N}^T$, $\boldsymbol{N}_- = \boldsymbol{N} - \boldsymbol{N}^T$, and $\boldsymbol{N} = \boldsymbol{N}_1$ or $\boldsymbol{N} = \boldsymbol{N}_1 \otimes \boldsymbol{N}_2$, $(\cdot)^T$ is the transpose of matrix $(\cdot)$. Then, we get iterative matrix of (5) as follows

$$\boldsymbol{M} = \left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_-\right)^{-1}\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)^{-1}\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_-\right).$$

Now what we need to prove is that the iterative sequence $\{\boldsymbol{P}^k\}$ of control vertexes converges to the unique solution $\boldsymbol{P}^*$, i.e., $\rho(\boldsymbol{M}) < 1$.

Theorem 1. *The two iterative processes of (3) and (4) are convergent, if the bases $\{N_j(u)\}_{j=0}^n$ and $\{S_j(v)\}_{j=0}^m$ are totally positive and their collection matrices $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are non-singular.*

Proof. Based on the similarity in-variance of spectral radius, the symmetric matrix $\boldsymbol{N}_+$, and the anti-symmetric matrix $\boldsymbol{N}_-$, we have

$$\begin{aligned}\rho(\boldsymbol{M}) =& \rho\left(\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)^{-1}\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_-\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_-\right)^{-1}\right)\\ \le& \left\|\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)^{-1}\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_-\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_-\right)^{-1}\right\|_2\\ \le& \left\|\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)^{-1}\right\|_2\left\|\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_-\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_-\right)^{-1}\right\|_2.\end{aligned}$$

$$\begin{aligned}&\because \boldsymbol{N}_-^T = -\boldsymbol{N}_-\\ &\therefore \left(\left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}\right)^T\left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}\\ &= \left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)^{-1}\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}\\ &= \left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)^{-1}\left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}\\ &= \boldsymbol{I}\end{aligned}$$

$\therefore \left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}$ is a unitary matrix, i.e., $\left\|\left(\boldsymbol{I} - \frac{\boldsymbol{N}_-}{2}\right)\left(\boldsymbol{I} + \frac{\boldsymbol{N}_-}{2}\right)^{-1}\right\|_2 = 1$.

Thus, $\rho(\boldsymbol{M}) \le \left\|\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}_+\right)\left(\boldsymbol{I} + \frac{1}{2}\boldsymbol{N}_+\right)^{-1}\right\|_2 = \max\limits_{\lambda_i \in \lambda(\boldsymbol{N}_+/2)}\left|\frac{1-\lambda_i}{1+\lambda_i}\right|$. From Lemma 1, we know $\lambda_i > 0, i = 0, 1, \cdots, n$, so $\rho(\boldsymbol{M}) < 1$. This completes the proof. □

4 Iterative Format of WHPIA

Similar to weighted PIA, we can accelerate the iterative process of the HPIA in a weighted approach. Thus, (3) and (4) can be written in two weighted forms respectively as follows

$$\begin{cases} \boldsymbol{P}_i^k = \boldsymbol{P}_i^{k-1/2} + \omega_1 \boldsymbol{\Delta}_{1i}^k, \boldsymbol{P}_i^{k-1/2} = \boldsymbol{P}_i^{k-1} + \omega_1 \boldsymbol{\Delta}_{2i}^k \\ \boldsymbol{P}_{ij}^k = \boldsymbol{P}_{ij}^{k-1/2} + \omega_2 \boldsymbol{\Delta}_{1ij}^k, \boldsymbol{P}_{ij}^{k-1/2} = \boldsymbol{P}_{ij}^{k-1} + \omega_2 \boldsymbol{\Delta}_{2ij}^k \end{cases}.$$

And their matrix forms can be obtained from (5)

$$\begin{cases} \left(\boldsymbol{I} + \frac{1}{2}\omega \boldsymbol{N}_-\right) \boldsymbol{P}^k = \left(\boldsymbol{I} - \frac{1}{2}\omega \boldsymbol{N}_+\right) \boldsymbol{P}^{k-1/2} + \omega \boldsymbol{P}^0 \\ \left(\boldsymbol{I} + \frac{1}{2}\omega \boldsymbol{N}_+\right) \boldsymbol{P}^{k-1/2} = \left(\boldsymbol{I} - \frac{1}{2}\omega \boldsymbol{N}_-\right) \boldsymbol{P}^{k-1} + \omega \boldsymbol{P}^0. \end{cases} \tag{6}$$

The iterative matrix of (6) is as follows

$$\boldsymbol{M} = \left(\boldsymbol{I} + \frac{1}{2}\omega \boldsymbol{N}_-\right)^{-1} \left(\boldsymbol{I} - \frac{1}{2}\omega \boldsymbol{N}_+\right) \left(\boldsymbol{I} + \frac{1}{2}\omega \boldsymbol{N}_+\right)^{-1} \left(\boldsymbol{I} - \frac{1}{2}\omega \boldsymbol{N}_-\right).$$

From Theorem 1, we know $\rho(\boldsymbol{M}) \leq \max\limits_{\lambda_i \in \lambda(\boldsymbol{N}_+/2)} \left|\frac{1-\omega\lambda_i}{1+\omega\lambda_i}\right|, \lambda_i > 0, i = 0, 1, \cdots, n, \omega > 0$ i.e., $\rho(\boldsymbol{M}) < 1$. Thus, the iterative process (6) is convergent.

Theorem 2. *Given two non-singular collocation matrices, $\boldsymbol{N}_1 = (N_j(u_i))_{i,j=0}^{n,n}$, $\boldsymbol{N}_2 = (S_j(V_i))_{i,j=0}^{m,m}$, which are defined on two NTP bases $\{N_j(u)\}_{j=0}^n$, $\{S_j(v)\}_{j=0}^m$. The HPIA with weight has the approximate fastest convergence rate when*

$$\omega^* = \frac{2}{\sqrt{\lambda_{max}(\boldsymbol{N} + \boldsymbol{N}^T)\lambda_{min}(\boldsymbol{N} + \boldsymbol{N}^T)}},$$

where $\boldsymbol{N} = \boldsymbol{N}_1$ or $\boldsymbol{N} = \boldsymbol{N}_1 \otimes \boldsymbol{N}_2$.

Proof. It is proved in [14] that the optimal spectral radius is

$$\frac{1}{\omega^*} = \sqrt{\lambda_{max}\left(\frac{\boldsymbol{N} + \boldsymbol{N}^T}{2}\right) \lambda_{min}\left(\frac{\boldsymbol{N} + \boldsymbol{N}^T}{2}\right)},$$

thus,

$$\omega^* = \frac{2}{\sqrt{\lambda_{max}(\boldsymbol{N} + \boldsymbol{N}^T)\lambda_{min}(\boldsymbol{N} + \boldsymbol{N}^T)}}.$$

□

5 Simulation

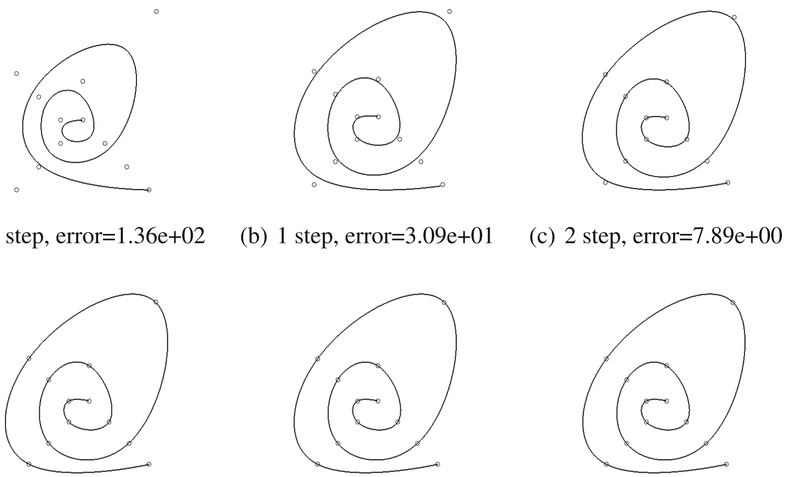
In this section, two examples are used for simulation to demonstrate the effectiveness of the proposed methods HPIA and WHPIA, and to make a simple comparison with PIA [6,7] and WPIA [10]. First, we give two test examples as follows, those are, an example of iterative curve interpolation and an example of iterative surface interpolation.

Example 1. 12 data points are taken in plane to constitute a 1×12 sequence:

$$((165,150)\ (75,150)\ (75,225)\ (170,265)\ (150,165)\ (90,165)\ (90,210)\ (120,220)\ (135,180)\ (105,180)\ (105,195)\ (120,195))$$

Example 2. 48 data points are taken in space to constitute a 7×9 matrix:

$$\begin{pmatrix} (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,-8,-10) & (8,-8,-10) & (8,0,-10) & (8,8,-10) & (0,8,-10) & (-8,8,-10) & (-8,0,-10) & (-8,-8,-10) & (0,-8,-10) \\ (0,-10,0) & (10,-10,0) & (10,0,0) & (10,10,0) & (0,10,0) & (-10,10,0) & (-10,0,0) & (-10,-10,0) & (0,-10,0) \\ (0,-15,10) & (15,-15,10) & (15,0,10) & (15,15,10) & (0,15,10) & (-15,15,10) & (-15,0,10) & (-15,-15,10) & (0,-15,10) \\ (0,-6,30) & (6,-6,30) & (6,0,30) & (6,6,30) & (0,6,30) & (-6,6,30) & (-6,0,30) & (-6,-6,30) & (0,-6,30) \\ (0,-6,50) & (6,-6,50) & (6,0,50) & (6,6,50) & (0,6,50) & (-6,6,50) & (-6,0,50) & (-6,-6,50) & (0,-6,50) \\ (0,-8,-55) & (8,-8,-55) & (8,0,-55) & (8,8,-55) & (0,8,-55) & (-8,8,-55) & (-8,0,-55) & (-8,-8,-55) & (0,-8,-55) \end{pmatrix}$$

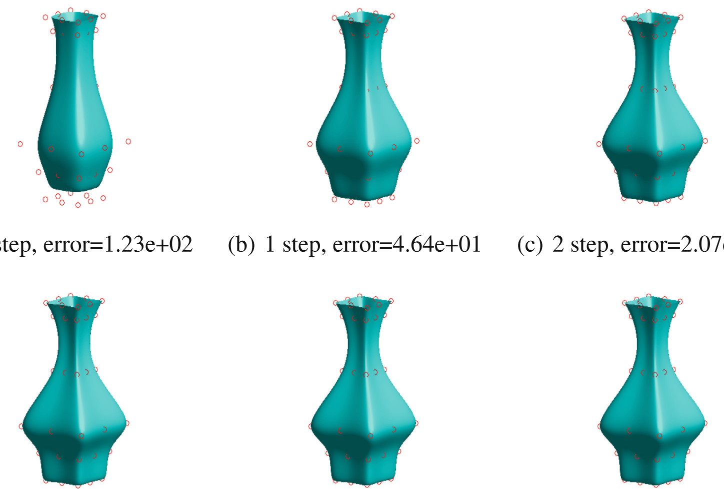


(a) 0 step, error=1.36e+02 (b) 1 step, error=3.09e+01 (c) 2 step, error=7.89e+00

(d) 4 step, error=9.79e-01 (e) 8 step, error=3.72e-02 (f) 16 step, error=1.27e-04

Fig. 1. HPIA iterative interpolation of curve example

Here, we choose B-spline to verify the effectiveness of HPIA and WHPIA, there are two reasons: on the one hand, B-spline has many excellent properties in expressing shapes; on the other hand, B-spline bases are NTP bases. We adopt cubic non-uniform B-spline and use centripetal parameterization method to calculate parameters of data points. The internal knots are determined by parameters of data points. The fitting error at each iteration level is taken as the total Euclidean norms of the adjusting vectors.



(a) 0 step, error=1.23e+02 (b) 1 step, error=4.64e+01 (c) 2 step, error=2.07e+01

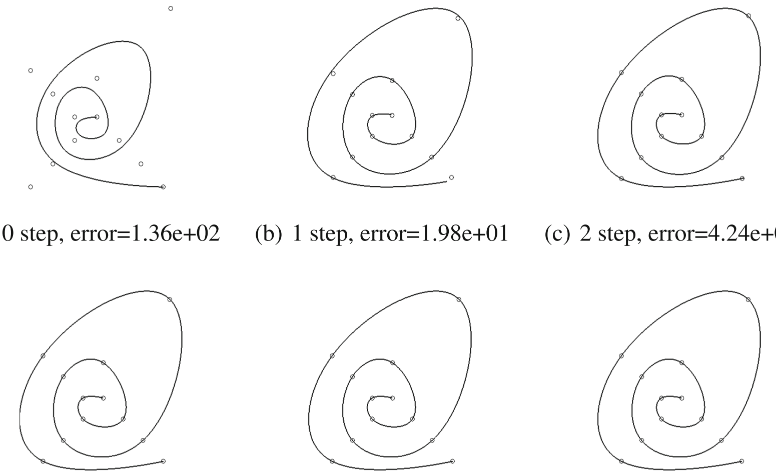
(d) 4 step, error=5.49e+00 (e) 8 step, error=9.88e-01 (f) 16 step, error=8.48e-02

Fig. 2. HPIA iterative interpolation of surface example

$$\begin{cases} \varepsilon\_curve_k = \sum\limits_{i=0}^{n} \|\mathbf{\Delta}_i^k\| \\ \varepsilon\_surface_k = \sum\limits_{i=0}^{n} \sum\limits_{j=0}^{m} \|\mathbf{\Delta}_{ij}^k\|. \end{cases}$$

The experimental results of the method proposed in this paper are shown in Figs. 1, 2, 3 and 4, where subfigure (a) represents the initial state of iteration, and (b)–(f) respectively illustrate the results after 1, 2, 4, 8 and 16 iterations, and the corresponding iteration error is attached to below the corresponding subfigure. It can be seen that the error of the weighted HPIA at the same iteration level is much smaller than that of the unweighted HPIA, which exemplifies the validity of the weighted version WHPIA.

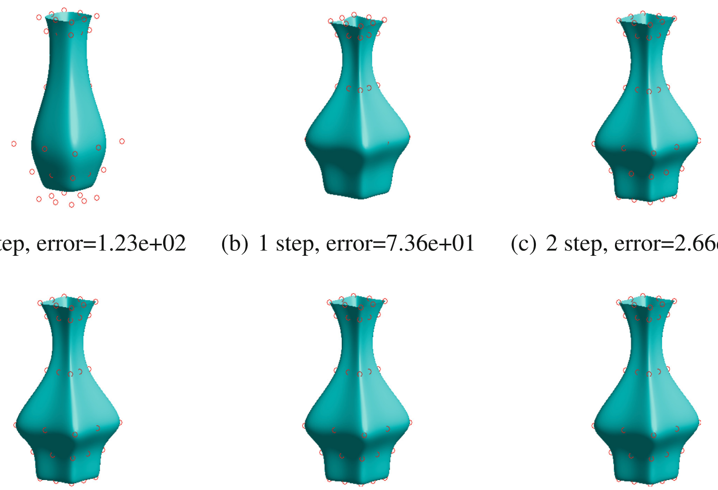
In addition, Figs. 5 and 6 illustrate the results of 16 iterations obtained by interpolating the data points in example 1 and example 2 using four methods, HPIA, WHPIA, PIA and WPIA. Due to the fact that the error of the later iteration is smaller, so to get a better look at the iterative effects, there are two parts of each figure, namely the first eight iterations and the next eight iterations. It can be seen that WHPIA performs best in Figs. 5 and 6, followed by HPIA. The examples used in this section are only intended to demonstrate the effectiveness of the methods presented by us. For other examples, we cannot guarantee that our approaches converge faster than PIA and WPIA because the spectral radius will vary with the NTP bases.



(a) 0 step, error=1.36e+02 (b) 1 step, error=1.98e+01 (c) 2 step, error=4.24e+00

(d) 4 step, error=2.38e-01 (e) 8 step, error=9.56e-04 (f) 16 step, error=1.86e-08

Fig. 3. WHPIA iterative interpolation of curve example



(a) 0 step, error=1.23e+02 (b) 1 step, error=7.36e+01 (c) 2 step, error=2.66e+01

(d) 4 step, error=2.72e+00 (e) 8 step, error=4.12e-02 (f) 16 step, error=2.69e-05

Fig. 4. WHPIA iterative interpolation of surface example

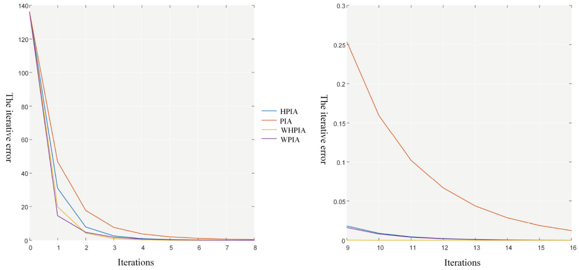


Fig. 5. Error comparison of iterative curve interpolation using four methods

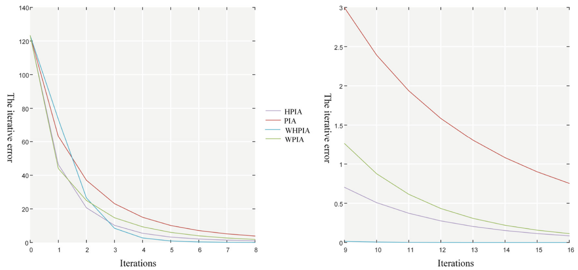


Fig. 6. Error comparison of iterative surface interpolation using four methods

6 Conclusion

In this paper, based on the HSS iterative method for solving linear equations, a new PIA approach called HPIA was proposed to solve the problems of curves and surfaces interpolation with normalized totally positive bases. And then, we weighted it to speed up the convergence rate of the iterative process, namely WHPIA, and gave the approximate value of the fastest convergence weight. Experimental results show that HPIA and WHPIA are effective in progressive iterative approximation of curves and surfaces. However, due to various NTP bases, it is not clear which method in WPIA and WHPIA has a smaller spectral radius, so it is impossible to prove theoretically which method has the fastest convergence speed.

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