A New MCMC Particle Filter Resampling Algorithm Based on Minimizing Sampling Variance

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Abstract. In order to solve the problem of particle divergence caused by deviation of sample distribution before and after resampling, a new Markov Chain Monte Carlo (MCMC) resampling algorithm based on minimizing sampling variance is proposed. First, MCMC transfer in which Particle Swarm Optimization (PSO) is possessed as the transfer kernel to construct Markov Chain is applied to the impoverished sample to combat sample degeneracy as well as sample impoverishment. Second, the algorithm takes the weighted variance as the cost function to measure the difference between the weighted particle discrete distribution before and after the resampling process, and optimizes the previous MCMC resampling by the minimum sampling variance criterion. Finally Experiment result shows that the algorithm can overcome particle impoverishment and realize the identical distribution of particles before and after resampling.

Keywords: PF-resampling · MCMC · PSO · Minimizing sampling variance

1 Introduction

To combat the inevitable weight degeneracy caused by SIS [1,2], SIR [3,4]viewed as a combination between SIS and resampling procedure was proposed. The basic idea of re-sampling is to copy the large weight particles according to the size of the weight values, and replace the small weight particles with the offspring of the large weight particles. The essence is to redistribute the weights so that more particles get sampling opportunities, which is based on sacrificing the diversity

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of samples. This means that, after numerous times of iterations, a large number of particles in the sample are only concentrated on a few particular points in the state space. This phenomenon is called "sample impoverishment" [3,4], mainly because of the strong correlation between particles, which is not sufficient to describe the randomness of the target posterior distribution. In order to obtain the sufficient sample diversity, balance between the proposal distribution and the real target posterior distribution should be paid more attention. In another word, more particles have the chance to be resampled as they are assigned weights that cannot be ignored. The related research on maintaining diversity of samples is described explicitly in reference [5], which is not overstated here. It is worth mentioning that a new MCMC resampling strategy was employed in [5], where PSO considered as transition kernels of MCMC had been applied to each particles so that all particles, theoretically, could be adjusted to the high likelihood areas in state-space instead of merely multiplying particles with high weights. In the former case, the basic idea of MCMC resampling algorithm is, after a sufficient burn-in time, constructing a Markov Chain reaching a stationary distribution, which is approximate to target posterior distribution. However, the resampled particles can not guarantee the unbiased estimation of the real target posterior. To combat sample impoverishment, a large number of resampling strategies including PSO-MCMC mentioned above, hybird resampling [6–8], but not limited to, adopt resampling from alternative sampling sets rather than original sets. Eventually, particle filters have to suffer from side effects of these biased re-sampling strategies. This means that, SIR sample impoverishment as well as deviation of sample distribution before and after resampling will both affect the estimation accuracy of samples to real target posterior, which could eventually lead to the divergence of the filter. Therefore, it is an unavoidable problem in sample estimation to verify the deviation of samples after resampling, which is also the research content of this paper. In this paper, a new MCMC resampling strategy in terms of satisfying Minimum sampling variance (MSV) is proposed, in which the former PSO-MCMC resampling algorithm has been optimized. The MSV criterion [9] theoretically guarantee any sample subset can reach the minimum sampling variance on condition that the resampling process satisfies the optimal weight condition and the specific sample number. Identical distribution of samples after and before resampling acquired by MSV means that resampling will not drift estimation to real target posterior resulting from reducing the loss of information in the resampling process. As a tool for tracking the state of a dynamic system modeled by Bayesian Network, PF also could be employed as an estimator to predict network traffic [14,15].

2 Identical Distribution of Resampling and Relative Evaluation Methods

Compared with parametric filters, the advantage of PF is regarded to be complete approximation to target posterior distribution particularly in non-linear and non-gaussian state models. Therefore resampled particles are expected to approximate the original distribution as much as possible. That is to say, distribution of resampled particles should be similar to the original distribution so long as no other new observation considered, which is called identical distribution attribute of resampling. However, in fact, dissimilarity of particle distribution after and before resampling is inevitable whether in theory or in engineering practice. Accordingly, it is necessary to set up a reliable evaluation system for the deviation or even variation of posterior distribution resulted from resampling. In this system, the extent of deviation after resampling should be evaluated, that is, how much resampling is competent to keep the original distribution. Therefore, the identical distribution attribute is expected to be the basic principle of designing a resampling algorithm, and it is also required that the particles before and after resampling are suppose to meet it. Specifically, we introduce and compare several common metrics such as kullback-Leibler divergence [10] (K-L divergence), kolmogorov-smirnov statistic (K-S statistic), and MSV [9] to measure differences between two probability distributions in the same state space.

2.1 Kullback-Leibler Divergence

Relative entropy, also called Kullback-Leibler divergence (K-L divergence), is a measurement to describe the difference between two probability distributions such as $P(x)$ and $Q(x)$. Then the relative entropy of $P(x)$ and $Q(x)$ is as follows.

$$D(P||Q) = \sum (P(x)\log(P(x)/Q(x))) \tag{1}$$

In Eq. 1, $P(x)$ and $Q(x)$ represent the probabilistic distributions before and after resampling respectively, and $D(P||Q)$ provides a measure of the extent of distribution difference caused by resampling. The larger the K-L divergence between $P(x)$ and $Q(x)$ is, the lower the similarity is.

2.2 Kolmogorov-Smirnov Test

Kolmogorov-smirnov test (K-S test), also refered to kolmogorov-smirnov statistic, is a non-parametric probability distribution test that is used to measure whether a sample conforms to a certain probability distribution or to compare whether the two probability distributions are identical. In our case, the Kolmogorov-Smirnov test provides a distance between the empirical distribution functions of two samples such as $P(x)$ and $Q(x)$ that represent the posterior distribution after and before resampling respectively. The empirical distribution function F_{n} for the observation X_{i} is defined as Eq. 2.

$$F_x(x) = \frac{1}{n}\sum_{i-1}^{n} I_{|-\infty,x|}(X_i) \tag{2}$$

Where $I_{|-\infty,x|}(X_i)$ is the indicator function, equal to 1 if $X_{\mathrm{i}} \leq x$ and equal to 0 otherwise. The K-S statistic for a given cumulative distribution function $F(x)$ is described as Eq. 3.

$$D_n = \sup_x |F_p(x) - F_q(x)| \tag{3}$$

where $\sup_x$ is the supremum of the set of distances, $F_p(x)$ and $F_{\mathrm{q}}(x)$ represent empirical distribution functions of posterior distributions $P(x)$ and $Q(x)$ respectively and D_{n} measures the discrepancy of these posterior distributions caused by resampling. According to Glivenko–Cantelli theorem, if $P(x)$ and $Q(x)$ are identical, then D_{n} converges to 0 almost surely in the limit when n goes to infinity.

2.3 Minimum Sampling Variance

The number of particle resampling must be an integer, that is $N_{\mathrm{t}}^{(m)}$. Assuming that resampling is unbiased, the equation $E(N_{\mathrm{t}}^{(m)}) = Nw_t^{(m)}$ should be satisfied. Obviously, there is a difference between the number of resampling and its expected value. Furthermore, a higher-order moment has a better ability to describe distribution difference than a first-order moment. Accordingly, we define the sampling variance is equal to the square difference mean between the number of times of the particle resampling and its definition is shown in Eq. 4.

$$SV = \frac{1}{M}\sum_{\mathrm{m}=1}^{M}(N_{\mathrm{t}}^{\mathrm{m}} - Nw_t^{(m)})^2 \tag{4}$$

SV in Eq. 4, considered as a cost function, can provide an effective measurement method for testing the discrepancy between weighted particle discrete distribution before and after resampling.

The smaller the value of SV in Eq. 4 is, the better the identical distribution attribute of the resampling algorithm is. If and only if these two distributions are exactly the same, the value of KL distance, K-S test and sampling variance are zero. Therefore, the SV, KL and K-S test are consistent in terms of describing the discrepancy of posterior distribution. Apart from that, SV has the advantages in less computation time.

3 A New MCMC Resampling Strategy Optimized by MSV

In order to minimize the sampling variance in Eq. 4, that is, to minimize the distribution differences to the maximum extent, the weight of the resampled particles should be set to equivalent as shown in Eq. 5.

$$\tilde{w}_t^{(n)} = \frac{1}{N} \tag{5}$$

However, this condition, describing in Eq. 5, is only satisfied in the traditional resampling method instead of the combined resampling algorithm. In this paper, We quote the constraints of the optimal resampling algorithm as the optimal weight conditions from related reference [9]. Under this condition, the essence of resampling problem is equal to determine the sampling times of each particle.

This means that, if the optimal weight condition, defined in Eq. 5, is satisfied, Eq. 4 can provide the minimum sampling variance.

To combat the deviation of posterior distribution resulted from a certain resampling algorithm, we put forward a new resampling strategy, named MCMC(PSO)-MSV, that the MSV is employed to optimize the previous MCMC resampling strategy mentioned in [5].

The MCMC(PSO)-MSV mainly involves two processes. In the first stage, PSO considered as the transition kernel of MCMC is applied to each particle in order to move particles to the high likelihood area in state-space. Afterward, to reduce information loss prompted by the previous MCMC resampling, MSV should be adopted. For sample set $\{\mathrm{x_k^i}\}_{\mathrm{i=1,\cdots,N}}$, if the condition, $N_{eff} \le N_{th}$, is satisfied, then the MCMC(PSO)-MSV resampling algorithm will be applied. Specifically, the implementation of the algorithm is illustrated as follows.

Step1: sample set $\{\mathrm{x_k^i}\}_{\mathrm{i=1,\cdots,N}}$ adjusted by PSO.

- searching and determining P_{gbest}, the largest weight of particles, as well as P^i_{gbest}, the maximum of weights among its iteration history respectively,

$$P_{gbest} = F\max(\{\mu^{\mathrm{i}}_{\mathrm{k,n}}\}_{\mathrm{i=0,\cdots,N-M-1}})$$
$$P^{i}_{gbest} = F\max(\{\mu^{\mathrm{i}}_{\mathrm{k,n}}\}_{\mathrm{n=0,\cdots,J}})$$

  Where $\mu^i_{k,n}$ is the weight of $X^i_{k,n}$.
- For each particle $X^i_{k,n}$, updating its moving rate $V^i_{k,n}$ and state,

$$V^i_{k,n+1} = V^i_{k,n} + \varphi_1(P^i_{pbest} - X^i_{k,n}) + \varphi_2(P_{gbest} - X^i_{k,n})$$
$$\hat{X}{}^i_{k,n+1} = X^i_{k,n} + V^i_{k,n+1}$$

in which φ_1 and φ_2 are random numbers subordinating Gauss distribution, and output of this process is a new set: $\{\hat{X}{}^i_{k,n+1}\}^{PSO}_{i=0,\cdots,N-M-1}$.

Step2: Metroplis-Hastings sampling (M-H sampling).

$$\alpha = \frac{P(\hat{X}{}^i_{k,n+1}\,|z_{1:k})q(X^i_{k,n};\hat{X}{}^i_{k,n+1})}{P(X^i_{k,n+1}\,|z_{1:k})q(\hat{X}{}^i_{k,n};X^i_{k,n+1})}$$

Where we generate a random number ρ, $\rho \sim u(0,1)$.

if $\rho \le \min(1,\alpha)$, then accept M-H sampling $X^i_{k,n+1} = \hat{X}{}^i_{k,n+1}$.
Else, then refuse M-H sampling $X^i_{k,n+1} = X^i_{k,n}$.

Step3: Judging convergence condition.
if $P_{gbest} \le \varepsilon$, then stop MCMC (PSO) process and move to next stage.

Step4: Each particle is resampled $MaxInteger(Nw^{(m)}_t)$ times, and the weight residuals, named $\hat{w}^{(m)}_t$, and number of particles produced in this process, named T, are represented respectively as follows.

$$\hat{w}_t^{(m)} = w_t^{(m)} - MaxInteger(Nw_t^{(m)})/N$$
$$T = \sum_{m=1}^{M} MaxInteger(Nw_t^{(m)})$$

Where Operator $MaxInteger(\cdot)$ provides the maximum integer.

Step5: For particular particles with relative larger value of $\hat{w}_t^{(m)}$ in top $N-T$ area, they will be resampled one more time. Specifically, we select $N-T$ elements with largest weight residuals from weight set $\{\mathrm{w}\wedge_t^{(m)}\}_{m=1}^{M}$, recorded as $MaxElement_{N-T}[\{\hat{w}_t^{(m)}\}_{\mathrm{m=1}}^{\mathrm{M}}]$, Where Operator $MaxElement_S[S]$ returns the largest element of set S.

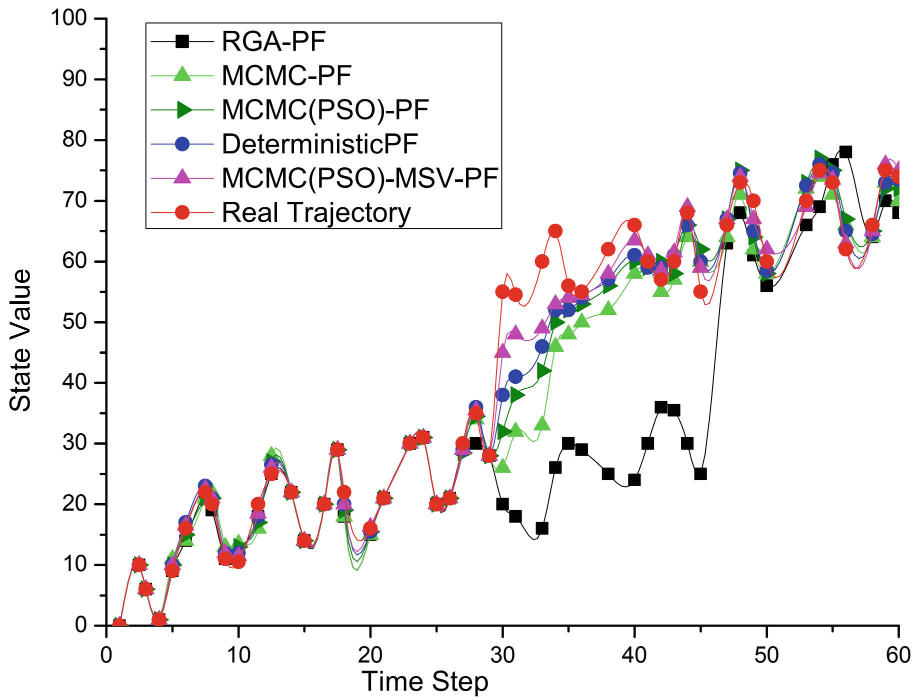

Fig. 1. True state VS filter estimation.

4 Experiment

In this paper, several resampling algorithms are numerically simulated with the classical model in [11], including an unbiased resampling, called deterministic resampling [12] (Deterministic-PF), genetic algorithm after residual resampling (RGA-PF) [13], MCMC resampling applying Metroplis-Hastings sampling (MCMC-PF), resampling with PSO as MCMC transaction kernel [5]

(MCMC(PSO)-PF), the new resampling strategy that MCMC(PSO)-PF optimized by minimum sampling variance (MCMC(PSO)-MSV-PF). Comparison will be carried out among these 5 resampling strategies in terms of estimation accuracy, sampling variance as well as RMSE. System model and observation model are presented as follows.

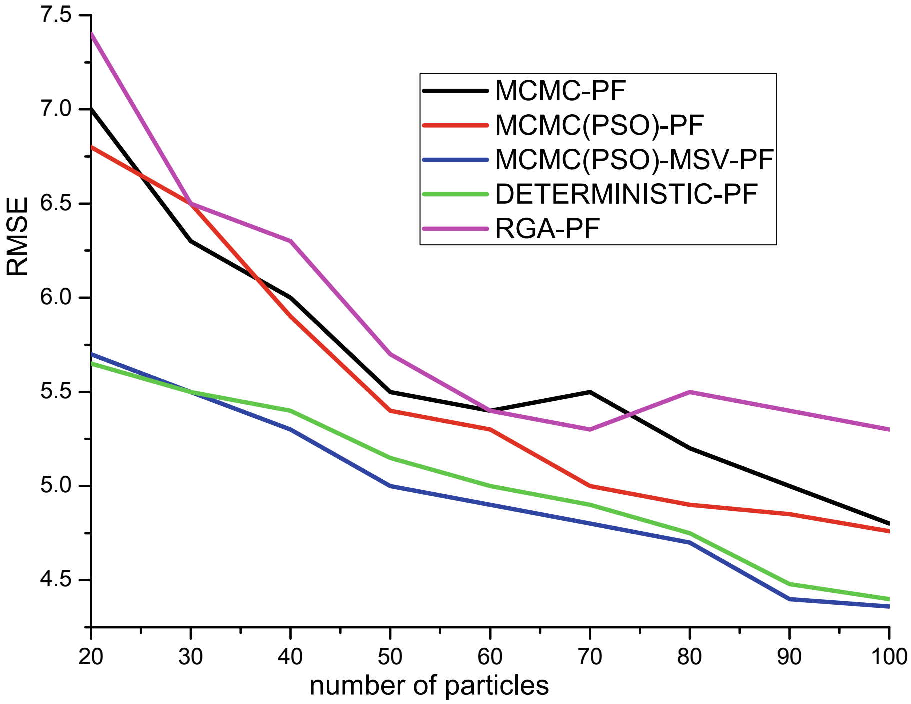

Fig. 2. Sampling variances of these 5 resampling strategies.

$$x_k = c_1 \cdot x_{k-1} + c_2 \cdot \frac{x_{k-1}}{1 + x_{k-1}^2} + c_3 \cdot \cos(1.2(k-1)) + \omega_k \tag{6}$$

$$y_k = \frac{x_k^2}{20} + \upsilon_k \tag{7}$$

in which x_k and y_k represent system state and observation at t time respectively; $c_1 = 1$, $c_2 = 12$, $c_3 = 7$; ω_k and ν_k are state noise and observation noise from distribution $\omega_k \sim N(0, \sigma_\omega^2)$, $\nu_k \sim N(0, \sigma_v^2)$ ($\sigma_\omega = \sigma_v = 2$) respectively.

In this case the number of Monte Carlo simulation, T, is 60, N_s means number of particles, $N_s = 200$ and x_0 represent initial state value, $x_0 = 0$. The state transition probability, $p(x_{t+1} \,|x_t\,)$, is applied as the proposal distribution to realize the state prediction.

In the experiment, the identical condition of sample detection is chosen as the resampling condition, that is, if $Neff \leq N_s/3$, PF enters resampling process.

Comparison of True states of targets and estimating states is shown as Fig. 1, while Fig. 2 demonstrates sampling variances of these 5 resampling strategies. SV is chosen as the unbiased evaluation strategy after resampling on account of its less expensive computing. RMSE (average mean square error) is also presented in Fig. 3 in which the number of particles has changed from 20 to 100. These figures indicate that, if the resampling algorithm satisfies the unbiased or asymptotically unbiased conditions, different resampling algorithms will obtain approximate estimation accuracy, especially when N_0 is greater than 90. However, the resampling algorithms with better identical distribution attributes, such as Deterministic-PF and MCMC(PSO)-MSV-PF, have obvious advantages in RMSE evaluation only when $N_0 \leq 45$. This means that whether the sample deviates from the original distribution after resampling has a greater influence on the estimation accuracy of the small sample set.

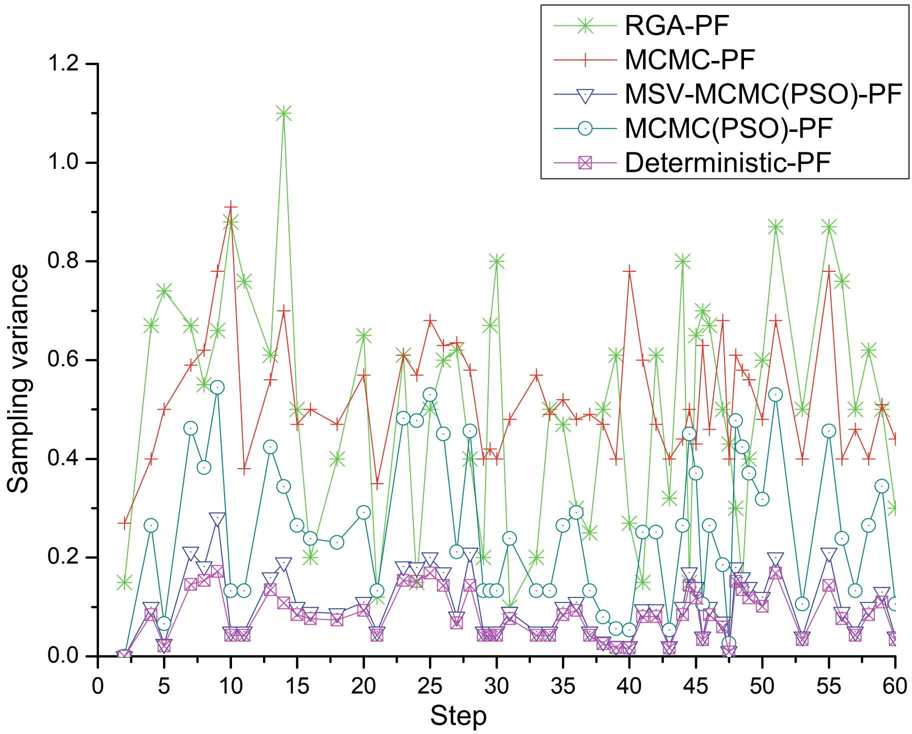

Fig. 3. RMSE comparison of these 5 algorithms as the number of particles changes from 20 to 100.

Besides, resampling algorithms such as MCMC-PF, RGA-PF and MCMC(PSO)-PF, are biased, which their sampling variances are shown in Fig. 2 respectively. Accordingly their RMSEs are obviously worse than that of decisive resampling and MSV-PF, as shown in the Fig. 3. This also shows the validity of sampling variance as unbiased evaluation of resampling algorithm, this means that whether the resampling algorithm is unbiased will affect the estimation accuracy. However, some resampling algorithms satisfy the unbiased condition, shown in formula 5, at the expense of sample diversity.

In Fig. 1, When the target state occurred a strong jump at $t = 30$, the Deterministic-PF diverged stemming from the loss of the diversity of samples which resulted in the filter losing ability to estimate the target posterior. While the MCMC(PSO)-MSV-PF is able to achieve a trade-off between preserving the diversity of samples and the identical distribution attributes. Consequently even coming accross the strong jump of target state, the MCMC(PSO)-MSV-PF can also estimate a posterior distribution close to the real target.

5 Conclusions

In order to solve the problem of particle divergence caused by particles deviation after resampling, in this paper, a new MCMC resampling strategy based on satisfying Minimum sampling variance (MSV) is proposed, in which the former PSO-MCMC resampling algorithm has been optimized. Identical distribution of samples after and before resampling acquired by MSV means that resampling will not drift estimation to real target posterior thanks to reducing the loss of information in the resampling procedure. The simulation result shows that the MCMC(PSO)-MSV-PF is superior to its counterparts in terms of preserving the diversity of samples and acquiring the identical distribution attributes before and after resampling.

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