Accelerated Matrix Inversion Approximation-Based Graph Signal Reconstruction

Qian Dang, Yongchao Wang(✉), and Fen Wang

Collaborative Innovation Center of Information Sensing and Understanding, State Key Laboratory of ISN, Xidian University, Xi'an 710071, Shaanxi, China
qdang@stu.xidian.edu.cn;ychwang@mail.xidian.edu.cn

Abstract. Graph signal processing (GSP) is an emerging field which studies signals lived on graphs, like collected signals in a sensor network. One important research point in this area is graph signal reconstruction, i.e., recovering the original graph signal from its partial collections. Matrix inverse approximation (MIA)-based reconstruction has been proven more robust to large noise than the conventional least square recovery. However, this strategy requires the K-th eigenvalue of Laplacian operator $\boldsymbol{\mathcal{L}}$. In this paper, we propose an efficient strategy for approximating the K-th eigenvalue in this GSP filed. After that, the MIA reconstruction method is modified by this proposed substitution, and thereby accelerated. Consequently, we apply this modified strategy into artificial graph signal recovery and real-world semi-supervised learning field. Experimental results demonstrate that the proposed strategy outperforms some existed graph reconstruction methods and is comparable to the MIA reconstruction with lower numerical complexity.

Keywords: Graph signal processing · Graph reconstruction
Semi-supervised learning

1 Introduction

With massive production of irregularly structured signals, graph signal processing (GSP) becomes an overwhelming research filed, which intends to extend classical discrete signal processing tools into graph signal domain [1,2]. Recently developed GSP technologies, like graph-based filtering, sampling and reconstruction on graphs, have been applied into various real-life data analysis, such as transportation network monitoring, semi-supervised learning and recommendation systems [3–5].

Graph signal reconstruction attempts to recovery a smooth graph signal from its partial observed samples (noiseless or corrupted). A noiseless bandlimited

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graph signal can be perfectly recovered by its samples if the sample size is larger than its bandwidth [6]. Authors in [7,8] attempted to find out the analogous Nyquist-Shannon sampling theorem for graph signals and proposed some instructive graph sampling and reconstruction strategies. Least square (LS) reconstruction was investigated in [9] for designing a sampling condition for unique recovery. However, the classical LS method requires full eigen-pair decomposition of the graph Laplacian operator and large matrix inversion. Authors in [10] proposed an iterative least squares reconstruction (ILSR) algorithm based on *projection on convex sets theorem* for detouring above complex computations. A generalized ILSR algorithm was proposed in [11] based on frame theory, which has faster convergence rate compared to conventional ILSR algorithm. Authors in [12] designed a robust graph signal recovery strategy via truncated Neumann series, termed as matrix inverse approximation (MIA) reconstruction, which approximates the LS solution but without full eigen-decomposition or matrix inversion. Another approach for recovering a graph signal is regularization, which makes use of the inherited smoothness property of graph signals [13].

In this paper, we focus on designing an efficient strategy for accelerating the existed MIA reconstruction which needs the K-th eigenvalue of Laplacian operator $\boldsymbol{\mathcal{L}}$. We propose an approximation method for computing this K-th eigenvalue in the GSP filed. Then, the MIA reconstruction method is improved by using this proposed substitution. In the sequence, we evaluate the performance of this method by applying it to artificial graph signal recovery and real-world semi-supervised learning field. Simulation results show that the proposed strategy is superior to some existing methods and is comparable to the MIA reconstruction with lower numerical complexity.

The remainder of paper is organized as follows. We first provide the preliminary background and MIA-based reconstruction strategy in Sect. 2 and propose the accelerated MIA graph signal reconstruction algorithm based on fast eigenvalue approximation in Sect. 3. Section 4 presents simulation results. Conclusions are presented in Sect. 5.

2 Notation and Background

A graph is represented as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathbf{W}\}$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of nodes and edges, and weight $\mathbf{W}(i,j) = w_{i,j}$ on edge $(i,j) \in \mathcal{E}$ represents the similarity between node i and node j. The degree matrix $\mathbf{D}$ is defined by $\mathbf{D} = \text{diag}(\mathbf{d})$ in which $d_i = \Sigma_j w_{i,j}$. Then, the normalized graph Laplacian matrix can be written as $\boldsymbol{\mathcal{L}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$, which is adopted as the variation operator in this paper [14]. $\boldsymbol{\mathcal{L}}$ is a symmetric and positive semi-definite matrix whose eigenvalues and eigenvectors are $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N \leq 2$ and $\mathbf{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_N\}$ respectively. A graph signal is a function $f : \mathcal{V} \to \mathbb{R}$, which can also be represented as a vector $\mathbf{f} \in \mathbb{R}^N$, where each element represents the function value on its corresponding node. Analogous graph Fourier transform (GFT) of a signal $\mathbf{f}$ is defined as $\hat{\mathbf{f}} = \mathbf{U}^T\mathbf{f}$ and the inverse GFT is $\mathbf{f} = \mathbf{U}\hat{\mathbf{f}}$. A signal is called bandlimited when there exists a number $K \in \{1, ..., N\}$ so that its GFT satisfies

$\hat{\mathbf{f}}_i = 0$, for all $i > K$, and the smallest K is called *bandwidth* of a graph signal $\mathbf{f}$. Graph signals with bandwidth at most K are called K-bandlimited (K-BL) graph signals and can be expressed as $\mathbf{f} = \mathbf{U}_K \hat{\mathbf{f}}_K$, where $\mathbf{U}_K$ means the first K columns of $\mathbf{U}$. We use $\mathcal{S}^c$ to denote the complementary set of $\mathcal{S}$. A restriction of a matrix $\mathbf{A}$ to rows in set $\mathcal{S}_1$ and columns in set $\mathcal{S}_2$ is denoted by the submatrix $\mathbf{A}_{\mathcal{S}_1\mathcal{S}_2}$. $\mathbf{A}_{\mathcal{S}\mathcal{S}}$ is abbreviated as $\mathbf{A}_{\mathcal{S}}$. Moreover, we use $\mathcal{S}_u$ to represent a uniqueness set [15] in this paper, and $\mathcal{S}_r$ to represent a random sampling set. $\mathbf{I}$ is a unit matrix whose dimension is determined by context.

Graph sampling is defined as a linear mapping $\mathbf{f}_{\mathcal{S}} = \mathbf{C}\mathbf{f}$, in which the sampling operator $\mathbf{C} \in \mathbb{F}^{M \times N}$ is [16]

$$\mathbf{C}_{ij} = \begin{cases} 1, & j = \mathcal{S}_i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $\mathcal{S}_i$ is the i-th sampling index, $\mathbb{F}^{M \times N}$ is the set of sampling operators corresponding to all sampling set $\mathcal{S}$ such that $|\mathcal{S}| = M$.

2.1 Matrix Inversion Approximation-Based Reconstruction Strategy

A sampled K-BL graph signal can be written as $\mathbf{f}_{\mathcal{S}} = \mathbf{C}\mathbf{U}_{\mathrm{K}}\hat{\mathbf{f}}_{\mathrm{K}}$ based on previously introduced notations. In noiseless condition, if $\mathrm{rank}(\mathbf{C}\mathbf{U}_{\mathrm{K}}) = \mathrm{K}$, a unique and perfect reconstruction $\tilde{\mathbf{f}}$ can be obtained via the LS solution [17]

$$\tilde{\mathbf{f}} = \mathbf{U}_K(\mathbf{C}\mathbf{U}_{\mathrm{K}})^{\dagger}\mathbf{f}_{\mathcal{S}} \tag{2}$$

where $(\cdot)^{\dagger}$ denotes the pseudo-inverse operator.

According to proposition 1 in [12], this LS solution is equal to

$$\begin{aligned} \tilde{\mathbf{f}} &= \mathbf{U}_K[(\mathbf{C}\mathbf{U}_K)^T\mathbf{C}\mathbf{U}_K]^{-1}(\mathbf{C}\mathbf{U}_K)^T\mathbf{f}_{\mathcal{S}} \\ &= \mathbf{U}_K \sum_{l=0}^{\infty} [\mathbf{I} - (\mathbf{C}\mathbf{U}_K)^T(\mathbf{C}\mathbf{U}_K)]^l \mathbf{U}_K^T\mathbf{C}^{\mathrm{T}}\mathbf{f}_{\mathcal{S}} \end{aligned} \tag{3}$$

After a series of derivations of (3), authors in [12] proposed the MIA reconstruction strategy which requires neither full eigen-decomposition nor matrix inversion:

$$\tilde{\mathbf{f}} = \mathbf{T}_{\mathcal{V}\mathcal{S}}\tilde{\mathbf{\Gamma}}\mathbf{f}_{\mathcal{S}} \tag{4}$$

where $\mathbf{T} = \mathbf{U}_K\mathbf{U}_K^T$ and $\tilde{\mathbf{\Gamma}} = \sum_{l=0}^{L}(\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}})^l$.

The ideal low-pass graph filter $\mathbf{T}$ has a kernel function as follows

$$h(\lambda) = \begin{cases} 1, \lambda \le \lambda_K \\ 0, \lambda > \lambda_K \end{cases} \tag{5}$$

$h(\lambda)$ can be approximated by a truncated Chebyshev polynomial, thereby $\mathbf{T}$ will be approached by $\mathbf{T}^{\mathrm{Ploy}} = \sum_{i=1}^{N}\left(\sum_{j=0}^{p}\beta_j\lambda_i^j\right)\boldsymbol{u}_i\boldsymbol{u}_i^T = \sum_{j=0}^{p}\beta_j\boldsymbol{\mathcal{L}}^j$ without the information of $\mathbf{U}_K$. It is clear that λ_K is required for realizing the

approximation of $\mathbf{T}$, since it is the cut-off frequency of this ideal low-pass filter. Actually, the complexity for calculating λ_K is $\mathcal{O}(N^3)$ in general. If Locally Optimal Block Prec-conditioned Conjugate Gradient (LOBPCG) method [18] is adopted for obtaining λ_K, its complexity is $\mathcal{O}\left(\left(|\mathcal{E}|\, M + TM^3\right) T_1\right)$ [14]. Authors in [12] stated that λ_K can be computed via a series of fast algorithms, under which its complexity will be $\mathcal{O}(RN)$, where $K < R \ll N$. In order to reduce the complexity for obtaining λ_K and simplify the caculation steps, we propose an efficient strategy in this GSP field to approximate λ_K.

3 Accelerated MIA Graph Signal Reconstruction Based on Fast Eigenvalue Approximation

In this section, we propose a simple method to calculate λ_K of $\boldsymbol{\mathcal{L}}$ in the GSP field based on the characteristics of cut-off frequency introduced in [19]. After that, we use the approximated eigenvalue to modify the conventional MIA reconstruction strategy.

3.1 Simple Strategy for Approximating λ_K of $\boldsymbol{\mathcal{L}}$ in the GSP Field

A set $\mathcal{S}_u \subset \mathcal{V}$ is called a uniqueness set for the space $PW_\omega(G)$, any signal $\mathbf{f} \in PW_\omega(G)$ can be perfectly reconstructed from its noiseless samples $\mathbf{f}_{\mathcal{S}_u}$ [15]. Based on this definition and the work achieved in [8], the cut-off frequency of $\mathcal{S}_u$ can be estimated as

$$\omega_c(\mathcal{S}_u) = \lim_{j\to\infty} \Omega_j(\mathcal{S}_u) \approx \Omega_k(\mathcal{S}_u) \triangleq (\sigma_{1,k})^{1/k} \tag{6}$$

where $\Omega_k(\mathcal{S}_u)$ denotes an estimator of the ideal cut-off frequency and $\sigma_{1,k}$ denotes the smallest eigenvalue of the submatrix $(\boldsymbol{\mathcal{L}}^k)_{\mathcal{S}_u^c}$. It is obvious that $\Omega_k(\mathcal{S}_u)$ tends to provide a better estimate with larger k, while the complexity will correspondingly increase. Moreover, as claimed in [19], the exact $\omega_c(\mathcal{S}_u)$ actually can be obtained by characterizing the uniqueness set in a different way, and the following theorem is presented in that paper.

Theorem 1. ([19]) *For a graph $\mathcal{G}$ with normalized Laplacian $\boldsymbol{\mathcal{L}}$ with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ and corresponding eigenvectors $\boldsymbol{u}_1, ..., \boldsymbol{u}_N$,the cut-off frequency of a subset of nodes $\mathcal{S}_u$ is given by*

$$\omega_c(\mathcal{S}_u) = \max\{\lambda_i : \dim\mathcal{N}[\boldsymbol{u}_1, ..., \boldsymbol{u}_i, \boldsymbol{e}_j : j \in \mathcal{S}_u^c] = 0\} \tag{7}$$

Hence $\mathcal{S}_u$ is a uniqueness set for $PW_\omega(\mathcal{G})$ if and only if $\omega \leq \omega_c(\mathcal{S}_u)$.

Let $\mathcal{S}_u = \{1, ..., N\} - \{j_1, ..., j_{N-K}\}$, then $\{\boldsymbol{e}_j : j \in \mathcal{S}_u^c\} = \{\boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}\}$. According to Steinitz exchange lemma, combined with the property that $\boldsymbol{u}_1, ..., \boldsymbol{u}_K$ are linearly independent, we can always find $\boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}$ from standard basis of $\mathbb{R}^N$ to make $\{\boldsymbol{u}_1, ..., \boldsymbol{u}_K, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}\}$ a basis for $\mathbb{R}^N$. Based on

theorem that for a matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$, $\text{rank}(\mathbf{A}) + \dim\mathcal{N}(\mathbf{A}) = \text{N}$, when $i = K$, we can derive

$$\begin{aligned}&\dim\mathcal{N}[\boldsymbol{u}_1, ..., \boldsymbol{u}_i, \boldsymbol{e}_j : j \in \mathcal{S}_u^c] \\ &= K + (N - K) - \text{rank}(\boldsymbol{u}_1, ..., \boldsymbol{u}_K, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}) \\ &= 0\end{aligned}$$

This actually means $\omega_c(\mathcal{S}_u) \geq \lambda_K$ based on Theorem 1. In the same way, when $i = K + 1$, we will have

$$\begin{aligned}&\dim\mathcal{N}[\boldsymbol{u}_1, ..., \boldsymbol{u}_i, \boldsymbol{e}_j : j \in \mathcal{S}_u^c] \\ &= K + 1 + (N - K) - \text{rank}(\boldsymbol{u}_1, ..., \boldsymbol{u}_{K+1}, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}) \\ &= 1 \neq 0\end{aligned}$$

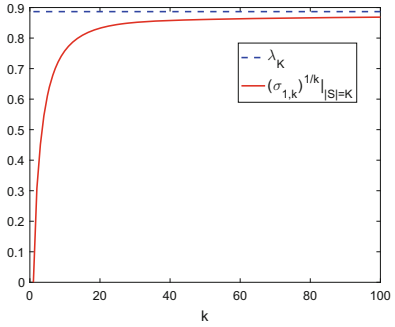


Fig. 1. Experimental result for evaluating equation (9), the estimated cut-off frequency of a random set $\mathcal{S}_r$ with $|\mathcal{S}_r| = 50$ and the exact solution λ_{50} of L.

which implies $\omega_c(\mathcal{S}_u) < \lambda_{K+1}$.

From Theorem 1, we know $\omega_c(\mathcal{S}_u)$ must be one eigenvalue of $\boldsymbol{\mathcal{L}}$. Moreover, we have clarified that $\omega_c(\mathcal{S}_u) \geq \lambda_K$ and $\omega_c(\mathcal{S}_u) < \lambda_{K+1}$, thus leading to an important investigation in the GSP field that $\omega_c(\mathcal{S}_u) = \lambda_K|_{|\mathcal{S}_u|=K}$. In fact, it is empirically illustrated that picking randomly $\boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}$ will always lead $\{\boldsymbol{u}_1, ..., \boldsymbol{u}_K, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_{N-K}}\}$ to be a basis of $\mathbb{R}^N$, which implies that a random sampling set $\mathcal{S}_r = \{1, .., N\} - \{j_1, ..., j_{N-K}\}$ can also get this conclusion. Therefore, the exact solution of the cutoff frequency of a random set $\mathcal{S}_r$ with $|\mathcal{S}_r| = K$ can be solved by

$$\omega_c(\mathcal{S}_r) = \lambda_K|_{|\mathcal{S}_r|=K} \tag{8}$$

Equations (6) and (8) actually present an excellent method to compute λ_K in this GSP field. After combining (6) and (8), we propose the following corollary:

Corollary 1. *For a graph $\mathcal{G}$ with normalized Laplacian $\boldsymbol{\mathcal{L}}$ whose eigenvalues are $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$, its K-th eigenvalue λ_K can be approximate by*

$$\lambda_K \approx (\sigma_{1,k})^{1/k}|_{|\mathcal{S}_r|=K} \tag{9}$$

where $\mathcal{S}_r \subset \mathcal{V}$ is randomly selected set with $|\mathcal{S}_r| = K$, $\sigma_{1,k}$ denotes the smallest eigenvalue of the reduced matrix $(\boldsymbol{\mathcal{L}}^k)_{\mathcal{S}_r^c}$. k is an estimation parameter which controls the trade-off between estimation accuracy and computational complexity, as exhibited in (6).

In this paragraph, we perform a numerical experiment to illustrate Corollary 1. We randomly generate a weighted graph with 500 nodes, where the connection probability between each pair is 0.3 and the weights on edges are randomly and independently generated from 0 to 1. Then we randomly select a sample set $\mathcal{S}_r$ such that $|\mathcal{S}_r| = 50$. With the increase of k, the approximate and exact solutions of the eigenvalue converge to a same value. As shown in Fig. 1, the larger k will make the $(\sigma_{1,k})^{1/k}$ closer to λ_{50}.

Algorithm 1 Outline of the proposed A-MIA reconstruction algorithm

Input: Graph variance operator $\boldsymbol{\mathcal{L}}$, observed signal $\mathbf{f}_{\mathcal{S}}$, bandwidth K, parameters L and k

Output: Reconstructed graph signal $\tilde{\mathbf{f}}$

1: Randomly select K nodes to constitute a set $\mathcal{S}_r$
2: Approximate λ_K of $\boldsymbol{\mathcal{L}}$ by $\lambda_K = (\sigma_{1,k})^{1/k}|_{|\mathcal{S}_r|=K}$
3: Calculate the truncated Chebyshev polynomial coefficients of $h_a(\lambda)$ in (11) and then compute $\mathbf{T}_a^{\text{Ploy}} = \sum_{j=0}^{p} \beta_j \boldsymbol{\mathcal{L}}^j$
4: Compute $\tilde{\boldsymbol{\Gamma}}_a = \sum_{l=0}^{L} \left[\mathbf{I}_{\mathcal{S}} - (\mathbf{T}_a^{\text{Ploy}})_{\mathcal{S}}\right]^l$
5: Return $\tilde{\mathbf{f}} = (\mathbf{T}_a^{\text{Ploy}})_{\mathcal{V}\mathcal{S}} \tilde{\boldsymbol{\Gamma}}_a \mathbf{f}_{\mathcal{S}}$

3.2 A-MIA Reconstruction Algorithm and Complexity Analysis

As we discussed in Sect. 2.1, $\lambda_K(\boldsymbol{\mathcal{L}})$ is required to approximate the ideal low-pass filter $\mathbf{T}$. As discovered in Corollary 1, we can estimate $\lambda_K(\boldsymbol{\mathcal{L}})$ efficiently by Eq. (9). According to (5) and (9), the kernel function can be approximated as

$$h_a(\lambda) = \begin{cases} 1, \lambda \leq (\sigma_{1,k})^{1/k}|_{|\mathcal{S}_r|=K} \\ \\ 0, \lambda > (\sigma_{1,k})^{1/k}|_{|\mathcal{S}_r|=K} \end{cases} \tag{10}$$

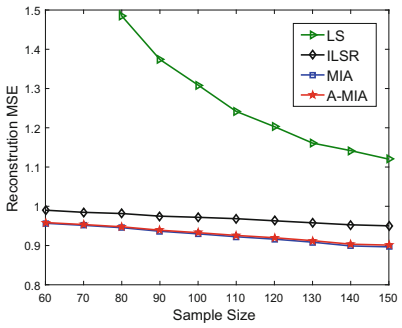
where $h_a(\lambda)$ denotes accelerated estimation of the original $h(\lambda)$.

With this new kernel function $h_a(\lambda)$, we can compute its corresponding Chebyshev matrix polynomial $\mathbf{T}_a^{\text{Poly}}$ more efficiently, since the calculation of λ_K is much simpler than the exact computation. Combining (4) and (10), we formally propose the accelerated MIA (A-MIA) reconstruction and illustrate the details of this modified MIA strategy in Algorithm 1.

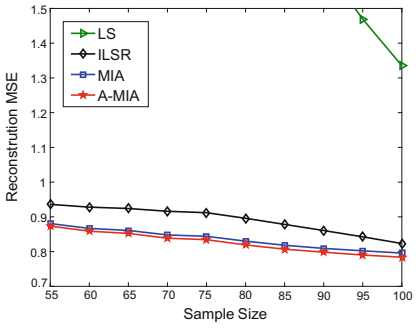
$$\tilde{\mathbf{f}} = (\mathbf{T}_a^{\text{Ploy}})_{\mathcal{V}\mathcal{S}} \tilde{\boldsymbol{\Gamma}}_a \mathbf{f}_{\mathcal{S}} \tag{11}$$

where $\tilde{\mathbf{\Gamma}}_a = \sum_{l=0}^{L} [\mathbf{I}_{\mathcal{S}} - (\mathbf{T}_a^{\text{Ploy}})_{\mathcal{S}}]^l$.

If the bandlimited graph signal is recovered by LS method, as we can see from Eq. (2), the eigenvector matrix $\mathbf{U}$ is required whose computation complexity is $\mathcal{O}(N^3)$ in general. While the conventional ILSR method doesn't involve eigenpair decomposition, it is essentially an iterative algorithm whose complexity and performance depends on the required steps for convergence. Both the MIA and the A-MIA algorithm just need to compute λ_K of $\boldsymbol{\mathcal{L}}$. The computational complexity of λ_K in the MIA method is $\mathcal{O}(RN)$ via a series of fast algorithms, where $K < R \ll N$. As described in subsection D in section IV of paper [14], the complexity of computing $(\sigma_{1,k})^{1/k}|_{|\mathcal{S}_r|=K}$, that is $\lambda_{\min}\left[(\boldsymbol{\mathcal{L}}^k)_{\mathcal{S}_r^c}\right]$, is $\mathcal{O}(kN)$ from the Rayleigh quotient perspective, where $k < K$ in general. In experiments, we set $k = 20$ and $K = 50$, which means the A-MIA algorithm is theoretically faster than the MIA method. Hence, we can safely claim that we propose a faster and simplified way for getting λ_K. The performance of this estimator compared to the exact one will be demonstrated in experiments.



(a) Reconstruction MSE in $\boldsymbol{G}1$.



(b) Reconstruction MSE in $\boldsymbol{G}2$.

Fig. 2. Simulation results for different reconstruction strategies where graph signals are all sampled randomly.

4 Experiments

In this section, we conduct some experiments to evaluate the efficiency and performance of the proposed reconstruction strategy. All experiments were performed in Matlab R2016a, running on a PC with Intel Pentium(R) 2.9 GHZ CPU and 8 GB RAM.

4.1 Artificial Graphs and Graph Signals

For artificial data-related simulations, we use the following models [14]:

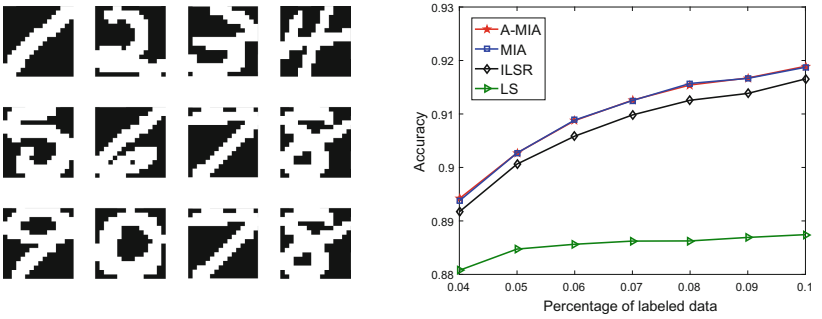
Artificial graphs: ($\boldsymbol{G}1$) Erdös-Renyi random graph (unweighted) with 1000

nodes and connection probability 0.01; ($\boldsymbol{G}2$) Unweighted Watts-Strogatz 'small world' model [20] with 1000 nodes, degree 8 and rewriting probability $\rho = 0.1$.

Artificial signals: The true signal is noise-free and approximately bandlimited with an exponentially decaying spectrum. The spectrum GFT coefficients are randomly generated from $\mathcal{N}(1, 0.5^2)$, followed by using the following filter to rescale $h(\lambda) = \begin{cases} 1, & \lambda \leq \lambda_K \\ e^{-4(\lambda - \lambda_K)}, & \lambda > \lambda_K \end{cases}$, where we choose $K = 50$.

Other Parameters: L is set to 10 and k is fixed at 20. The Chebyshev function in the GSP-toolbox package [21] is adopted to realize the Chebyshev polynomials approximation, where p = 10 and $\alpha = 8$. Random sampling is used for all reconstruction strategies.

As we analyzed in Sect. 3.2, the complexity of the proposed A-MIA algorithm is theoretically lower than the conventional MIA algorithm. Moreover, experimental results depicted in Fig. 2a, b demonstrate that the A-MIA algorithm is superior to the LS and ILSR algorithm and achieves almost the same performance as the MIA algorithm with lower complexity in both $\boldsymbol{G}1$ and $\boldsymbol{G}2$.



(a) Pixel images of handwritten digits. (b) Classification accuracy comparison.

Fig. 3. Performance comparison for different reconstruction strategies on the handwritten digits dataset.

4.2 Application in the Semi-supervised Learning Field

We apply the proposed algorithm to a classification task on the USPS handwritten digits dataset [22]. This dataset consists of 1100 pixel images of size 16×16 for each digit 0 to 9. We randomly select 100 samples from dataset for each digit to form a subset which will consist of 1000 feature vectors of dimension 256. For each instance, these selected feature vectors are used to constructed a symmetrized $\boldsymbol{\kappa}$-nearest neighbor graph via Gaussian kernel weighting function $\mathbf{W}(i,j) = \begin{cases} \exp\left(-\frac{[\mathrm{dist}(i,j)]^2}{2\sigma^2}\right), & \text{if } \mathrm{dist}(i,j) \leq \boldsymbol{\kappa} \\ 0, & \text{otherwise} \end{cases}$, where $\mathrm{dist}(i,j) = \|\mathbf{f}_i - \mathbf{f}_j\|_2^2$, and $\mathbf{f}_i$ is the feature vector composed of pixel intensity values of the i-th image.

We fix parameter $\sigma = 1$ and $\boldsymbol{\kappa} = 10$. The bandwidth K of the graph signal is approximately 50. We adopt the spectral proxies sampling algorithm [8] to select the nodes to label, based on which different recovery strategies are evaluated via reconstruction accuracy.

Figure 3a shows the pixel images of different handwritten digits. Figure 3b compares the classification accuracy of different reconstruction methods in terms of the percentages of labeled data. It depicts that our proposed method has higher accuracy than the conventional LS and ILSR algorithm and achieves almost the same performance as the original MIA reconstruction with theoretically lower complexity. Both the MIA and A-MIA reconstruction algorithms utilize an approximate low-pass filter $\mathbf{T}^{\mathrm{Ploy}}$ which has a slowly decaying spectral kernel. Therefore, they can catch more information of approximately bandlimited graph signals, thus leading to superior performance when applied into real-world tasks.

5 Conclusion

In this paper, we propose an efficient method for approximating the K-th eigenvalue of the Laplacian operator in the GSP filed, and modify the conventional MIA reconstruction strategy by the approximate eigenvalue. Compared with some existed methods, the modified strategy can achieve better performance in both artificial datasets and real-world semi-supervised tasks with lower complexity.

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