SPARSE SIGNAL SENSING WITH NON-UNIFORM UNDERSAMPLING AND FREQUENCY EXCISION

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ABSTRACT

We propose a novel compressive sensing algorithm for cognitive radio networks, based on non-uniform under-sampling. It is known that the spectrum of uniformly under-sampled signals exhibit frequency aliasing, whereby the frequency location is impossible. To alleviate aliasing, non-uniform sampling can be used. This, however, generates a high level of frequency leakage that prevents detection of weaker signals. To alleviate this problem, we introduce a novel iterative frequency excision technique that allows to detect tones or modulated signals below the original noise floor due to leakage. This method can be used in cognitive radio sensing engines, allowing to sense very wide bandwidths with a relatively low average sample rate. 20dB of leakage reduction can easily be achieved with this method.

Index Terms— spectrum sensing, cognitive radio, sub-Nyquist sampling, undersampling, frequency leakage, signal detection, compressive sensing.

1. INTRODUCTION

In the future, many cognitive radio networks may rely on spectrum sensing to monitor and coordinate the instantaneous use of the spectrum. Indeed, several authors and regulatory bodies have reported that the utilization of the radio spectrum by licensed and unlicensed wireless systems, e.g., TV broadcasting, cellular systems, wireless local area networks, etc .. is actually is quite low [1]. Specifically, the instantaneous usage of many frequency bands is very irregular. This has triggered the idea of reusing the unused portions of the spectrum by secondary users in an opportunistic way with cognitive radios (CR) [2], [3]. For CR to be able to reuse unused frequency bands, spectrum sensing is a necessity. Since large bands of several GHz may have to be sensed in the near future, sampling at the Nyquist rate can be prohibitive. Fortunately, because the spectrum is largely under-utilized, the signal to be sensed is sparse in the frequency domain. Compressive sensing exploits this sparsity.

In many implementations, it is desirable or even unavoidable to reduce the number of samples from the input signal without changing its characteristics. If the number of samples taken is reduced below the Nyquist rate, aliasing occurs and perfect reconstruction is no longer possible [4]. However, if the sampling instants are not taken at a constant rate or, in other words, if the sampling instants do not lie on a regular grid, it is still possible to reduce or completely avoid aliasing [5]. Perfect signal reconstruction is even possible under certain conditions, one of them stating that the average sampling rate must be higher than the so-called Landau-Nyquist rate (twice the sum of the bandwidths of all signals) and the sampling instants must be chosen in a special way [6] and [7], which is a challenging problem; hence, non-uniform random sampling is usually applied. However, doing so, significant leakage is created in the spectral estimate of the signal, which results in a degraded SNR.

When an undersampled signal must be reconstructed, the framework of compressive sensing [8] can be used. The central results of CS state that a sparse undersampled signal can be recovered by solving a convex (and rather complex) program. For detection or sensing, however, perfect reconstruction is not necessary. What is needed is that the spectrum estimate is sufficiently reliable to decide correctly about the presence or absence of signals. This is precisely the scope of this work: to analyze how tone or signal detection is possible based on samples of a signal that are taken much below the Nyquist rate. We convert the undersampled signal to the frequency domain where significant leakage masks the weaker signals, hence preventing their detection. We then rely on a novel frequency-domain excision technique to reduce gradually the frequency leakage, thereby making the detection of the weaker signals possible.

The structure of this paper is as follows: Section 2 describes the spectrum of an undersampled signal and Section 3 describes the non-uniform DFT estimators. Section 4 introduces the tone detection technique and simulation results are provided in Section 5. We conclude in Section 6.[1]

[1]Notational conventions: we use normal latin characters for time-domain signals (a) and tilde characters for frequency-domain signals ($\tilde{a}$); vectors and matrices are denoted by a single and double under-bar respectively ($\underline{a}$ and $\underline{\underline{A}}$); the superscripts ^{T} and ^{H} denote the matrix transpose and complex conjugate transpose respectively ($\underline{\underline{A}}^T$ and $\underline{\underline{A}}^H$); the superscript † is used to indicate the pseudo-inverse ($\underline{\underline{A}}^\dagger$).

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2. NON-UNIFORM SAMPLING - TIME AND FREQUENCY DOMAIN REPRESENTATIONS

Given a continuous-time signal $x_c(t)$, sampling is achieved by element-wise multiplication with a sampling function $s(t)$ which is a series of dirac pulses, yielding the sampled signal $x_s(t)$ and its frequency-domain counterpart $X_s(f)$:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_c(t)\delta(t - kT_S) \quad (1)$$

$$X_s(f) = \frac{1}{T_S}\sum_{k=-\infty}^{\infty} X_c(f - \frac{k}{T_S}). \quad (2)$$

In the case of non-uniform sampling, sampling occurs at times $t_s(k) = t_k$ that do not lie on a regular grid; we have the following time and frequency-domain representations:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_c(t)\delta(t - t_k) \quad (3)$$

$$X_s(f) = \frac{1}{T_S}\sum_{k=-\infty}^{\infty} x_c(t_k)e^{-j2\pi f t_k}. \quad (4)$$

The familiar form of the spectrum in (2) shows that the main spectrum is replicated at every integer multiples of the sampling rate $F_S = 1/T_S$. On the contrary, nothing can be said about the (non-)periodicity of the spectrum of a non-uniformly sampled signal as given in (4). The PSD of a non-uniformly sampled sequence has been studied by several authors [5], [9]. The main conclusions are as follows: the PSD contains both attenuated replicas and frequency leakage. If the observation time is long enough and the time instants t_k are sufficiently random, it can be shown that all spectral replicas disappear.

3. FROM THE CONVENTIONAL DFT TO THE NON-UNIFORM SUB-NYQUIST DFT

We will now illustrate the difference between conventional DFT-based spectral estimation and the non-uniform sub-Nyquist DFT. We define the following extended DFT for non-uniformly undersampled signals:

$$X(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x(t_k)e^{-j2\pi(t_k)(\frac{n}{N}F_S)}$$

$$n = -m\frac{N}{2}\cdots m\frac{N}{2} - 1. \quad (5)$$

Equation (5) simplifies to the conventional uniform, Nyquist sampled DFT (U-DFT) when $m = 1$ and $t_k = kT_S$ [2]. When $m > 1$ and $t_k = kT_S$, we have an extended uniform DFT

[2]We have chosen to define, without loss of generality, the DFT over the interval $[-m\frac{N}{2}\cdots m\frac{N}{2} - 1]$ rather than $[0 \ldots mN - 1]$

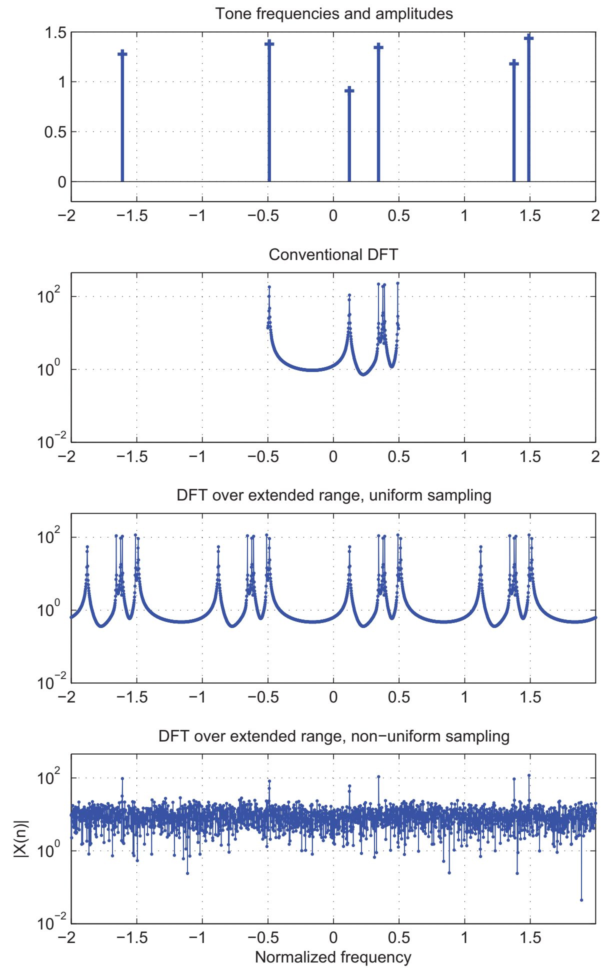

Fig. 1. DFT spectra for the example. (a) tones; (b) conventional uniform DFT; (c) uniform extended DFT; (d) non-uniform extended DFT

(U-eDFT), which corresponds to uniform undersampling. Finally, for $m > 1$ and $t_k \neq kT_S$, we have a non-uniform extended DFT (NU-eDFT). In this case, the average sampling rate $F_{S,av}$ is made equal to $F_S = 1/T_S$.

We illustrate the properties of these DFTs with an example. We generate a signal consisting of six complex exponentials at different frequencies within the interval $[-m\frac{N}{2}\cdots m\frac{N}{2} - 1]$, with $m = 4$. The tone magnitudes and frequencies and all DFTs are plotted in Figure 1. Figure 1b shows the conventional U-DFT that has a frequency range limited to $[-F_S/2, F_S/2]$ and aliases frequencies outside this range. The U-eDFT in Figure 1c does not provide additional information. Rather, the aliased spectrum of Figure 1b is repeated $m = 4$ times. The NU-eDFT as per (5) is plotted in Figure 1d. We observe that the aliasing is indeed eliminated and the amplitudes of the peaks are only approximately correct. In addition, there is a significant amount of *frequency leakage*. This phenomenon is actually one of the most severe limitations of non-uniform sub-Nyquist sampling because it can mask weak signals in the PSD.

4. TONE DETECTION

4.1. Basic Tone Detection

We consider a signal consisting of the sum of M complex exponentials (each defined by their complex amplitude w_i and frequency f_i) and additive complex white gaussian noise of variance σ_n^2. After sampling, this signal has the following time ($\underline{x}$) and frequency ($\underline{\tilde{x}}$) domain representation:

$$\underline{x} = \sum_{i=1}^{M} w_i e^{j(2\pi f_i \underline{t})} + \underline{n} \qquad (N \times 1) \tag{6}$$

$$\underline{\tilde{x}} = \underline{\underline{F}} \cdot \underline{x} \qquad (mN \times 1) \tag{7}$$

where $\underline{t}$ is the $(N \times 1)$ vector of the sampling time instants $[t_0 \dots t_{N-1}]$, $\underline{f}$ is the $(mN \times 1)$ vector of the frequency coordinates $[-m/2 \dots m/2 - (1/2N)]$ at which the spectrum is evaluated and $\underline{\underline{F}}$ is the (rectangular) NU-eDFT matrix

$$\underline{\underline{F}} = \frac{1}{\sqrt{N}} e^{-j(2\pi \underline{f} \cdot \underline{t}^T)} \qquad (mN \times N). \tag{8}$$

Our goal is to detect the presence of all the components of the signal with a reasonable probability of detection and probability of false alarm. The NU-eDFT for a signal consisting of 10 tones is shown in Figure 2, showing that a simple peak search on the magnitude will not allow to make a proper tone detection for all 10 tones: the weaker tones are buried in the noise-like leakage. To overcome this problem, we have developed two leakage reduction techniques relying on peak excision that are detailed in the following sections.

4.2. Peak Excision by frequency-domain peak search - Method 1

In order to allow detection of lower amplitude peaks, our method removes the stronger peaks together with the leakage that they create. The rationale for this is that stronger tones contribute the most to the leakage. Hence, once their frequency and complex amplitude is estimated, we can eliminate each tone from the received signal and continue searching for lower magnitude tones.

We start by detecting the strongest peak in the frequency spectrum, which provides an estimate of its frequency $\hat{f}_1$ and complex amplitude $\hat{w}_1$. The contribution of this tone to the global signal can be expressed in the time-domain as:

$$\underline{p}_1 = \hat{w}_1 e^{j2\pi \hat{f}_1 \underline{t}} = w_1 \underline{y}_1. \tag{9}$$

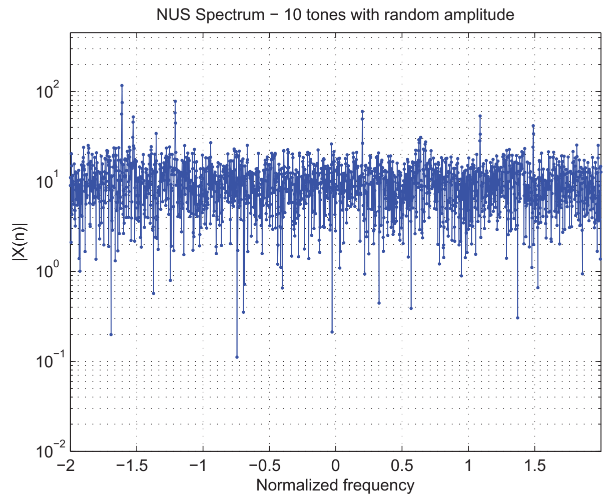

Fig. 2. Extended DFT spectrum with non-uniform sampling with $m = 4$. 10 tones were generated, some are masked by the leakage.

The signal after the first excision is described by:

$$\underline{x}_1 = \underline{x} - \underline{p}_1 \tag{10}$$

$$\underline{\tilde{x}}_1 = \underline{\underline{F}} \cdot (\underline{x} - \underline{p}_1) = \underline{\tilde{x}} - \underline{\underline{F}} \cdot \underline{p}_1 \tag{11}$$

which shows that the excision can be performed in either the time or frequency domain. This process can then be repeated with the second strongest peak and so on until some convergence criterion is met (see Section 4.4). Note that, in this method, the estimation of both the frequency $\hat{f}_j$ and the complex amplitude $\hat{w}_j$ are performed in the frequency domain.

4.3. Peak Excision by frequency-domain peak search and least-squares leakage minimization

4.3.1. *Excision of a single tone - Method 2*

In the previous method, the complex amplitude $\hat{w}_j$ is estimated in the frequency domain. Actually, because of the leakage generated by all the other signals on the currently detected peak, this complex amplitude estimation is degraded. Hence, we estimate the complex amplitude $\hat{w}_j$ by minimizing a least-squares criterion as follows. Assuming that the frequency $\hat{f}_j$ is correct, the leakage L_1 after excision of the first tone is

$$\begin{aligned} L_1 &= |\underline{\underline{A}} \cdot \underline{\tilde{x}}_1|^2 \\ &= \underline{x}_1^H \cdot \underline{\underline{G}} \cdot \underline{x}_1 \\ &= (\underline{x} - w_1 \underline{y}_1)^H \cdot \underline{\underline{G}} \cdot (\underline{x} - w_1 \underline{y}_1). \end{aligned} \tag{12}$$

where $\underline{\underline{G}} = \underline{\underline{F}}^H \cdot \underline{\underline{A}}^H \cdot \underline{\underline{A}} \cdot \underline{\underline{F}}$ and $\underline{\underline{A}}$ is an identity matrix except for some zeros at the indices corresponding to the frequency

$\hat{f}_j$ and its close neighboring frequencies (e.g. 3 frequency bins at each side). Those zeros avoid the peak due to $\underline{p}_1$ in the estimation of the leakage caused by $\underline{p}_1$ itself. The complex amplitude w_1 that minimizes the leakage, given $\hat{f}_1$, is the value of w_1 for which the derivative of L_1 is zero:

$$\frac{\partial}{\partial w_1}(L_1) = -\underline{x}^H \underline{\underline{G}} \underline{y}_1 + w_1^* \underline{y}_1^H \underline{\underline{G}} \underline{y}_1 = 0. \qquad (13)$$

The optimal value of w_1 follows directly:

$$w_1^* = \frac{\underline{x}^H \underline{\underline{G}} \underline{y}_1}{\underline{y}_1^H \underline{\underline{G}} \underline{y}_1}. \qquad (14)$$

This process is then repeated with the second strongest peak and so on until some convergence criterion is met (see Section 4.4).

4.3.2. *Excision of multiple tones at once - Method 3*

It is further possible to estimate jointly the complex amplitudes of multiple tones in a single operation if several peaks have been detected. The method developed in the previous section must be augmented with one dimension to handle multiple tones. Assuming we want to excise Q tones, we have

$$\underline{p} = \underline{\underline{Y}} \cdot \underline{w} \qquad (15)$$

where $\underline{\underline{Y}}$ is a matrix whose qth column is a complex exponential at frequency f_q and $\underline{w}$ is a vector containing the corresponding Q complex amplitudes w_q. The optimization problem can then be written as

$$\underline{\hat{w}} = arg \min_{\underline{w}} \; tr\{\underline{\underline{G}} \cdot (\underline{x} - \underline{\underline{Y}} \cdot \underline{w}) \cdot (\underline{x}^H - \underline{w}^H \cdot \underline{\underline{Y}}^H)\} \qquad (16)$$

where $\underline{\underline{G}} = \underline{\underline{F}}^H \cdot \underline{\underline{A}}^H \cdot \underline{\underline{A}} \cdot \underline{\underline{F}}$ as for the single tone case but $\underline{\underline{A}}$ is an identity matrix except for some zeros at the Q indices corresponding to the frequencies f_q and their close neighboring frequencies. The solution is obtained by setting all partial derivatives to zero:

$$\frac{\partial}{\partial \underline{w}}(L) = 0. \qquad (17)$$

After some lengthy calculus, the solution is found as:

$$\underline{w} = [\underline{\underline{Y}}^H \cdot \underline{\underline{G}} \cdot \underline{\underline{Y}}]^{-1} \cdot \underline{\underline{Y}}^H \cdot \underline{\underline{G}} \cdot \underline{x} \qquad (18)$$

which is a generalization of (14). This process is also repeated until some convergence (see Section 4.4).

4.3.3. *Comparison with previously known methods*

The technique proposed in this work should not be confused with the Matching Pursuit (MP) [10]. Indeed, when estimating the complex magnitudes of the detected signals, MP is not able to take into account the minimization of the frequency leakage introduced by the detected signals themselves. This is the key feature of our proposed frequency excision technique.

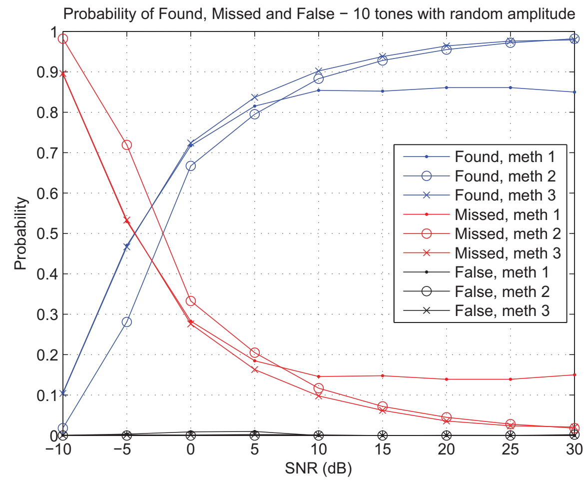

Fig. 3. Performance curves for Scenario 1. (Ten tones, m=4, Methods 1, 2 and 3).

4.4. Convergence criterion

To avoid false detections and false alarms, we must provide our excision scheme with two tests that are performed after each iteration:

1. The first test is a leakage reduction criterion. It consists in measuring the leakage power before and after an excision. As long as the excision reduces the leakage power, the iterations can continue. Otherwise, the last peak detection (which would result in a bad excision) is not used for excision and the process is halted. This test mostly avoids wrong detections after several peaks have been eliminated.

2. The second test is a "peak quality" criterion. It consists in estimating by how much the currently detected strongest peak is higher than the power of the rest of the spectrum. This test is useful for terminating the iterations but also to avoid starting iterations when noise only is present.

5. SIMULATION RESULTS

We have carried out some simulations to analyze the performance of the spectrum sensing with non-uniform undersampling and our excision schemes. We considered three scenarios:

Scenario 1: ten tones, m=4, Methods 1, 2 and 3. This corresponds to the detection of a few narrowband signals.

Scenario 2: three broadband signals that are each ten tones wide, m=4, Methods 1, 2 and 3. This corresponds to the sensing of modulated signals with moderate bandwidth, with a low spectrum usage.

Scenario 3: three signals, each two tones wide, and three signals, each 20 tones wide, m=4, Methods 1, 2 and 3. This

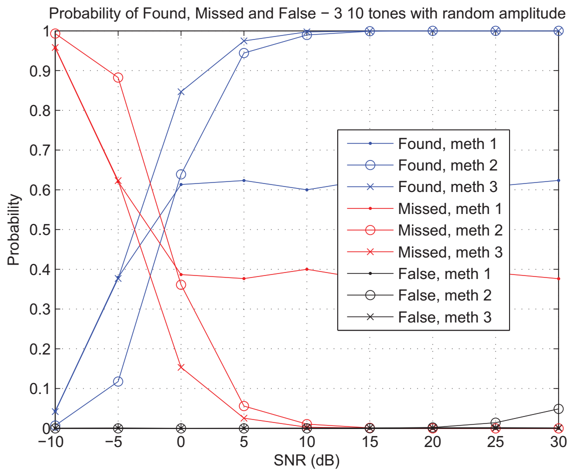

Fig. 4. Performance curves for Scenario 2. (Three broadband signals, each ten tones wide, m=4, Methods 1, 2 and 3).

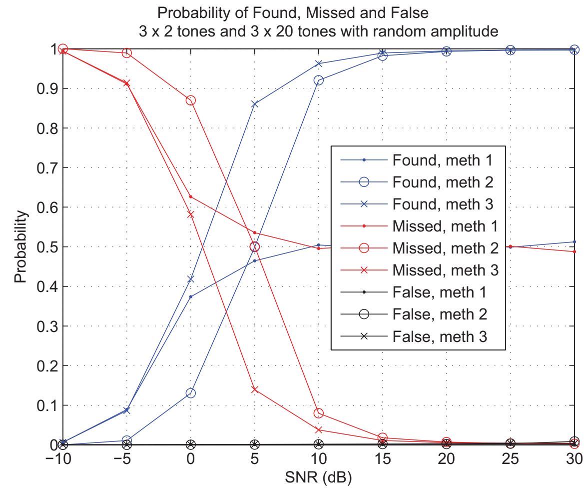

Fig. 5. Performance curves for Scenario 3. (Three signals, each two tones wide, and three signals, each 20 tones wide, m=4, Methods 1, 2 and 3).

corresponds to the sensing of a more ”crowded” environment with a mix of narrowband and broadband modulated signals.

The results for the three scenarios are shown in Figure 3, 4 and 5. They highlight the superior performance of the methods that minimize the frequency leakage (methods 2 and 3). When the number of signals increases, the detection probability of Method 1 starts to saturate because it does not cope well with the leakage. We can also observe that, thanks to the convergence criteria, the level of false detections for all methods is kept to a small value.

6. CONCLUSION

In this work, we have proposed a method to perform spectrum sensing on non-uniformly under-sampled signals. The non-uniform sampling is helpful because it allows to suppress the frequency domain aliasing that normally comes along with under-sampling. On the other hand, it introduces frequency leakage that masks weaker signals. Our contribution was to devise novel frequency excision methods that reduce the impact of this leakage. We have shown by simulation that our method is applicable to tones, narrowband and wideband signals and mixtures thereof; hence, it is useful for a wide range of spectrum sensing scenarios. This technique is especially attractive for the sensing of a wide RF band with a fast ADC that is driven much below the Nyquist rate to save power.

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