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Asymptotic Approximation of the Standard Condition Number Detector for Large Multi-Antenna Cognitive Radio Systems

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Abstract

Standard condition number (SCN) detector is a promising detector that can work efficiently in uncertain environments. In this paper, we consider a Cognitive Radio (CR) system with large number of antennas (eg. Massive MIMO) and we provide an accurate and simple closed form approximation for the SCN distribution using the generalized extreme value (GEV) distribution. The approximation framework is based on the moment-matching method where the expressions of the moments are approximated using bi-variate Taylor expansion and results from random matrix theory. In addition, the performance probabilities and the decision threshold are considered. Since the number of antennas and/or the number of samples used in the sensing process may frequently change, this paper provides simple form decision threshold and performance probabilities offering dynamic and real-time computations. Simulation results show that the provided approximations are tightly matched to relative empirical ones.

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1. Introduction

Cognitive Radio (CR), firstly proposed by Mitola [1], is the technology that provides solution for the scarcity and inefficiency in using the spectrum resource [2]. For the CR to operate efficiently and to provide the required improvement in spectrum efficiency, it must be able to effectively detect the presence/absence of the Primary User (PU). Thus, Spectrum Sensing (SS) is the key element for the presence/absence detection process in any CR guarantee.

Several SS techniques were proposed in the last decade [3], however, Eigenvalue Based Detector (EBD) has been shown to overcome noise uncertainty challenges and performs adequately even in low SNR conditions as it does not need any prior knowledge about the noise power or signal to noise ratio. EBD is based on the eigenvalues of the received signal covariance matrix and it utilises results from Random Matrix Theory (RMT) [4]. It detects the presence/absence of the PU by exploiting receiver diversity and includes the Largest Eigenvalue (LE) detector [5–7], the Scaled Largest Eigenvalue (SLE) detector [7–9] , and the Standard Condition Number (SCN) detector [5, 10–17].

Let $\boldsymbol{W}$ be the sample covariance matrix and denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$ its ordered eigenvalues; then the SCN of $\boldsymbol{W}$, defined as the ratio of the maximum to minimum eigenvalues, is given by:

$$X = \frac{\lambda_1}{\lambda_K}. \quad (1)$$

The SCN detector compares X with a certain threshold to decide about the existence of the PU. This threshold was set according to Marchenco-Pastur law (MP) in [5] which states that the largest eigenvalue (λ_1) and the smallest eigenvalue (λ_K) converge asymptotically to constants; however, this threshold is not related to any

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error constraints. In [10], the authors have provided an approximated relation between the threshold and the False-Alarm Probability (P_{fa}) by exploiting the Tracy-Widom distribution (TW) for the largest eigenvalue while maintaining the MP law for the smallest one. TW distribution is an asymptotic distribution for λ_1 [18]. This work was further improved in [11, 12] by considering the TW distribution and by using the Curtiss formula for the distribution of the ratio of random variables [19]. In these two cases, the threshold could not be computed online and Lookup Tables (LUT) should be used instead. The exact distribution of the SCN was, also, derived in [13] for 2 antennas and in [14] for 3 antennas, however, it is very complicated to extend this work for CR with higher number of antennas.

On the other hand, Massive MIMO (Multiple-Input-Multiple-Output) is viewed as one of the foundational 5G technologies [20]. Thus, CR with massive MIMO technology (equipped with tens to hundreds of antennas) could use any multi-antenna SS technique to identify the unused channels while achieving a significant increase of the performance of the SS detector. EBD presents an efficient way for multi-antenna SS and could be used efficiently in CR with Massive MIMOs. However, in such practical scenario, the number of antennas and/or the number of samples used in the sensing process may frequently change. Accordingly, the implementation of the decision threshold must be dynamic and rely on real-time computations and thus simple form of the performance probabilities and the decision threshold are required.

In this paper, we are interested in finding a simple approximation for the SCN detector's performance probabilities and decision threshold. This allows the system, equipped with tens to hundreds of antennas, to dynamically compute its threshold online according to the instantaneous scenario. For this purpose, we propose to asymptotically approximate the SCN distribution with the Generalized Extreme Value (GEV) distribution by matching the first three central moments. This approximation yields a simple and yet accurate closed form expression for the SCN detector. Accordingly, the threshold could be simply computed. The main contributions of this paper are summarized as follows:

- Derivation of the asymptotic central moments of the extreme eigenvalues.
- Derivation of an asymptotic approximated form of the central moments of the SCN from that of the extreme eigenvalues.
- Proposition of a simple and asymptotic closed form approximation for the SCN distribution using the central moments.
- Derivation of a simple form for the performance probabilities and the decision threshold for the SCN detector.

The rest of this paper is organized as follows. Section 2 provides the system model including the hypotheses analysis. Section 3 provides the asymptotic mean, variance and skewness of the extreme eigenvalues under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses. The asymptotic mean, variance and skewness of the SCN are derived in section 4. Then, we propose a new asymptotic approximation for the SCN distribution. In section 5, we derive the form of the performance probabilities as well as the decision threshold for the SCN detector. Theoretical findings are validated by simulations in section 6 while the conclusion is drawn in section 7.

Notations. Vectors and Matrices are represented, respectively, by lower and upper case boldface. The symbols $|.|$ and $tr(.)$ indicate, respectively, the determinant and trace of a matrix while $(.)^T$, and $(.)^\dagger$ are the transpose, and Hermitian symbols respectively. $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Symbols $\sim$ stands for "distributed as", $E[.]$ for the expected value and $||.||^2$ for the norm.

2. System Model

Consider a CR system equipped with K receiving antennas aiming to detect the presence/absence of a single PU during a sensing period corresponds to N samples. After collecting N samples from each antenna, the received signal matrix, $\boldsymbol{Y}$, is given by:

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & y_1(2) & \cdots & y_1(N) \\ y_2(1) & y_2(2) & \cdots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_K(1) & y_K(2) & \cdots & y_K(N) \end{pmatrix}, \quad (2)$$

where $y_k(n)$ is the baseband sample at antenna $k = 1 \cdots K$ and instant $n = 1 \cdots N$.

For this detection problem, two hypotheses exist: (i) $\mathcal{H}_0$: there is no PU and the received sample is only noise; and (ii) $\mathcal{H}_1$: the PU exists. The received vector, at instant n, under both hypotheses is given by:

$$\mathcal{H}_0 : \; y_k(n) = \eta_k(n), \quad (3)$$
$$\mathcal{H}_1 : \; y_k(n) = h_k(n)s(n) + \eta_k(n), \quad (4)$$

with $\eta_k(n)$ is a complex circular white Gaussian noise with zero mean and unknown variance σ_η^2, $h_k(n)$ is a the channel coefficient between the PU and antenna k at instant n, and $s(n)$ stands for the primary signal sample modeled as a zero mean Gaussian random variable with variance σ_s^2. Without loss of generality, we suppose that $K \leq N$ and the channel is considered constant during the sensing time for simplicity.

$\mathcal{H}_0$ hypothesis. By considering $\mathcal{H}_0$ hypothesis, the received samples are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2. Consequently, the sample covariance matrix is a central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ where K is the size of the matrix, N is the number of Degrees of Freedom (DoF), and $\sigma_\eta^2 \mathbf{I}_K$ is the correlation matrix.

$\mathcal{H}_1$ hypothesis. By considering $\mathcal{H}_1$ hypothesis and following our assumptions, the sample covariance matrix is a non-central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K, \mathbf{\Omega}_K)$ where $\mathbf{\Omega}_K$ is a rank-1 non-centrality matrix [21].

Let $\widehat{\Sigma}_K$ be the correlation matrix defined as:

$$\widehat{\Sigma}_K = \sigma_\eta^2 I_K + {}^{\Omega_K}/_N, \tag{5}$$

and denote by $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \cdots, \sigma_K]^T$ its vector of eigenvalues. Then $\boldsymbol{W}$, under $\mathcal{H}_1$, could be modeled as a central semi-correlated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$[22]. Since $\mathbf{\Omega}_K$ is a rank-1 matrix, then $\widehat{\Sigma}_K$ belongs to the class of spiked population model with all but one eigenvalue of $\widehat{\Sigma}_K$ are still equal to σ_η^2 while σ_1 is given by:

$$\sigma_1 = \sigma_\eta^2 + {}^{\omega_1}/_N, \tag{6}$$

with ω_1 is the only non-zero eigenvalue of $\mathbf{\Omega}_K$.

Denote the channel power by σ_h^2, then the average signal to noise ratio (SNR), under $\mathcal{H}_1$, is defined by:

$$\rho = \frac{\sigma_s^2 \sigma_h^2}{\sigma_\eta^2}, \tag{7}$$

where σ_s^2 could be estimated by $(\|\boldsymbol{s}\|^2/N)$, and the channel power $\sigma_h^2 = (\|\boldsymbol{h}\|^2/K)$. The non-centrality matrix is given by:

$$\mathbf{\Omega}_K = \Sigma_K^{-1} \boldsymbol{M}\boldsymbol{M}^\dagger = \frac{1}{\sigma_\eta^2} \|\boldsymbol{s}\|^2 \boldsymbol{h}\boldsymbol{h}^\dagger, \tag{8}$$

where Σ_K is the covariance matrix of $\boldsymbol{Y}$, defined as $\Sigma_K = E[(\boldsymbol{Y} - \boldsymbol{M})(\boldsymbol{Y} - \boldsymbol{M})^\dagger] = \sigma_\eta^2 I_K$, and $\boldsymbol{M}$ is the mean of $\boldsymbol{Y}$ defined as $\boldsymbol{M} = E[\boldsymbol{Y}] = \boldsymbol{h}\boldsymbol{s}^T$ with $\boldsymbol{h} = [h_1 h_2 \cdots h_K]^T$ and $\boldsymbol{s} = [s(1)s(2)\cdots s(N)]^T$.

As a result, and by using the property that the trace of a matrix equals the sum of its eigenvalues, then ω_1 could be written as:

$$\omega_1 = tr(\mathbf{\Omega}_K) = \frac{1}{\sigma_\eta^2} \|\boldsymbol{s}\|^2 tr(\boldsymbol{h}\boldsymbol{h}^\dagger) = NK\rho. \tag{9}$$

3. Assymptotic Moments of Extreme Eigenvalues

This section considers the statistical analysis of the extreme eigenvalues (λ_1 and λ_K) of the sample covariance matrix ($\boldsymbol{W}$) by considering both hypotheses. Since SCN is not affected by the noise power, let $\sigma_\eta^2 = 1$ and define the Asymptotic Condition (AC) and the Critical Condition (CC) as follows:

$$\textbf{AC}: \quad (K, N) \to \infty \text{ with } K/N \to c \in (0, 1), \tag{10}$$

$$\textbf{CC}: \quad \rho > \rho_c = \frac{1}{\sqrt{KN}}. \tag{11}$$

These conditions are important as will be stated through this paper.

3.1. $\mathcal{H}_0$ hypothesis

Let $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ be the maximum and minimum eigenvalue of $\boldsymbol{W}$ under $\mathcal{H}_0$ respectively, then:

Distribution of $\lambda_1^{\mathcal{H}_0}$. Denote the centered and scaled version of $\lambda_1^{\mathcal{H}_0}$ of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_1' = \frac{\lambda_1^{\mathcal{H}_0} - a_1(K, N)}{b_1(K, N)} \tag{12}$$

with $a_1(K, N)$ and $b_1(K, N)$, the centering and scaling coefficients respectively, are defined by:

$$a_1(K, N) = (\sqrt{K} + \sqrt{N})^2 \tag{13}$$

$$b_1(K, N) = (\sqrt{K} + \sqrt{N})(K^{-1/2} + N^{-1/2})^{\frac{1}{3}} \tag{14}$$

then, as AC is satisfied, λ_1' follows a TW distribution of order 2 (TW2) [23].

Distribution of $\lambda_K^{\mathcal{H}_0}$. Denote the centered and scaled version of $\lambda_K^{\mathcal{H}_0}$ of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_K' = \frac{\lambda_K^{\mathcal{H}_0} - a_2(K, N)}{b_2(K, N)} \tag{15}$$

with $a_2(K, N)$ and $b_2(K, N)$, the centering and scaling coefficients respectively, are defined by:

$$a_2(K, N) = (\sqrt{K} - \sqrt{N})^2 \tag{16}$$

$$b_2(K, N) = (\sqrt{K} - \sqrt{N})(K^{-1/2} - N^{-1/2})^{\frac{1}{3}} \tag{17}$$

then, as AC is satisfied, λ_K' follows a TW2[24].

Central Moments of $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$. The mean, variance and skewness of λ_1' and λ_K' are that of the TW2. They are given by $\mu_{TW2} = -1.7710868074$, $\sigma_{TW2}^2 = 0.8131947928$ and $\mathcal{S}_{TW2} = 0.2240842036$ respectively [25]. Accordingly, using (12), the mean, variance and skewness of $\lambda_1^{\mathcal{H}_0}$ are, respectively, given by:

$$\mu_{\lambda_1^{\mathcal{H}_0}} = b_1(K, N)\mu_{TW2} + a_1(K, N), \tag{18}$$

$$\sigma^2_{\lambda_1^{\mathcal{H}_0}} = b_1^2(K, N)\sigma_{TW2}^2, \tag{19}$$

$$\mathcal{S}_{\lambda_1^{\mathcal{H}_0}} = \mathcal{S}_{TW2}, \tag{20}$$

and using (15), the mean, variance and skewness of $\lambda_K^{\mathcal{H}_0}$ are, respectively, given by:

$$\mu_{\lambda_K^{\mathcal{H}_0}} = b_2(K,N)\mu_{TW2} + a_2(K,N), \tag{21}$$

$$\sigma^2_{\lambda_K^{\mathcal{H}_0}} = b_2^2(K,N)\sigma^2_{TW2}, \tag{22}$$

$$\mathcal{S}_{\lambda_K^{\mathcal{H}_0}} = -\mathcal{S}_{TW2}. \tag{23}$$

3.2. $\mathcal{H}_1$ hypothesis

Let $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ be the maximum and minimum eigenvalue of $\boldsymbol{W}$ under $\mathcal{H}_1$ respectively, then:

Distribution of $\lambda_1^{\mathcal{H}_1}$. Denote the centered and scaled version of $\lambda_1^{\mathcal{H}_1}$ of the central semi-correlated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$ by:

$$\lambda_1'' = \frac{\lambda_1^{\mathcal{H}_1} - a_3(K,N,\sigma)}{\sqrt{b_3(K,N,\sigma)}} \tag{24}$$

with $a_3(K,N)$ and $b_3(K,N)$, the centering and scaling coefficients respectively, are defined by:

$$a_3(K,N,\sigma) = \sigma_1(N + \frac{K}{\sigma_1 - 1}) \tag{25}$$

$$b_3(K,N,\sigma) = \sigma_1^2(N - \frac{K}{(\sigma_1 - 1)^2}) \tag{26}$$

then, as AC and CC are satisfied, λ_1'' follows a standard normal distribution ($\lambda_1'' \sim \mathcal{N}(0,1)$)[26].

Distribution of $\lambda_K^{\mathcal{H}_1}$. As mentioned in [27], when $\widehat{\Sigma}_K$ has only one non-unit eigenvalue such that CC is satisfied, then only one eigenvalue of $\mathbf{W}$ will be pulled up. In other words, and as it could be deduced from [28, Proof of Lemma 2], the remaining $K-1$ eigenvalues of $\mathbf{W}$ $(\lambda_2^{\mathcal{H}_1}, \cdots, \lambda_K^{\mathcal{H}_1})$ have the same distribution as the eigenvalues of $\mathbf{W} \sim \mathcal{CW}_{K-1}(N, \mathbf{I}_{K-1})$ under $\mathcal{H}_0$ hypothesis.

Denote the centered and scaled version of $\lambda_K^{\mathcal{H}_1}$ of the central semi-correlated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$ by:

$$\lambda_K'' = \frac{\lambda_K^{\mathcal{H}_1} - a_2(K-1,N)}{b_2(K-1,N)} \tag{27}$$

with $a_2(K,N)$ and $b_2(K,N)$ are, respectively, given by (16) and (17). Then, as the AC and CC are satisfied, λ_K'' follows a TW2.

It is worth mentioning that as CC is not satisfied and AC is satisfied, then $\lambda_1^{\mathcal{H}_1}$ follows TW2 distribution of $\lambda_1^{\mathcal{H}_0}$ [26]. Thus, non of the eigenvalue of $\mathbf{W}$ will be pulled up and $\lambda_K^{\mathcal{H}_1}$ follows TW2 distribution of $\lambda_K^{\mathcal{H}_0}$. Accordingly, the PU signal has no effect on the eigenvalues and could not be detected. It follows that the same analysis under $\mathcal{H}_0$ hypothesis is applied for this case.

Central Moments of $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$. The mean, variance and skewness of $\lambda_1^{\mathcal{H}_1}$ are, due to (24), given respectively by:

$$\mu_{\lambda_1^{\mathcal{H}_1}} = a_3(K,N,\sigma), \tag{28}$$

$$\sigma^2_{\lambda_1^{\mathcal{H}_1}} = b_3(K,N,\sigma), \tag{29}$$

$$\mathcal{S}_{\lambda_1^{\mathcal{H}_1}} = 0, \tag{30}$$

and using (27), the mean, variance and skewness of $\lambda_K^{\mathcal{H}_1}$ are respectively given by:

$$\mu_{\lambda_K^{\mathcal{H}_1}} = b_2(K-1,N)\mu_{TW2} + a_2(K-1,N), \tag{31}$$

$$\sigma^2_{\lambda_K^{\mathcal{H}_1}} = b_2^2(K-1,N)\sigma^2_{TW2}, \tag{32}$$

$$\mathcal{S}_{\lambda_K^{\mathcal{H}_1}} = -\mathcal{S}_{TW2}. \tag{33}$$

As a result, this section provides a simple form for the central moments of the extreme eigenvalues. These moments are used, in the next section, to derive an approximation for the mean, the variance and the skewness of the SCN under both hypotheses.

4. SCN Distribution Approximation

This section approximates the asymptotic distribution of the SCN by the GEV distribution using moment matching. First, we consider both detection hypotheses and we derive an approximation of the mean, the variance and the skewness of the SCN to be used in the next subsection for the approximation.

4.1. Asymptotic Central Moments of the SCN

The bi-variate first order Taylor expansion of the function $X = g(\lambda_1, \lambda_K) = \lambda_1/\lambda_K$ about any point $\theta = (\theta_{\lambda_1}, \theta_{\lambda_K})$ is written as:

$$X = g(\theta) + g'_{\lambda_1}(\theta)(\lambda_1 - \theta_{\lambda_1}) + g'_{\lambda_K}(\theta)(\lambda_K - \theta_{\lambda_K}) + O(n^{-1}), \tag{34}$$

with g'_{λ_i} is the partial derivative of g over λ_i.

Let $\theta = (\mu_{\lambda_1}, \mu_{\lambda_K})$ with μ_{λ_1} and μ_{λ_K} are the means of λ_1 and λ_K respectively, then it could be proved that:

$$E[X] = g(\theta), \tag{35}$$

$$\begin{aligned} E\left[(X - g(\theta))^2\right] = {} & g'_{\lambda_1}(\theta)^2 E\left[(\lambda_1 - \theta_{\lambda_1})^2\right] \\ & + g'_{\lambda_K}(\theta)^2 E\left[(\lambda_K - \theta_{\lambda_K})^2\right] \\ & + 2g'_{\lambda_1}(\theta)g'_{\lambda_K}(\theta)E\left[(\lambda_1 - \theta_{\lambda_1})(\lambda_K - \theta_{\lambda_K})\right], \end{aligned} \tag{36}$$

$$\begin{aligned} E\left[(X - g(\theta))^3\right] = {} & g'_{\lambda_1}(\theta)^3 E\left[(\lambda_1 - \theta_{\lambda_1})^3\right] \\ & + g'_{\lambda_K}(\theta)^3 E\left[(\lambda_K - \theta_{\lambda_K})^3\right] \\ & + 3g'_{\lambda_1}(\theta)^2 g'_{\lambda_K}(\theta)E\left[(\lambda_1 - \theta_{\lambda_1})^2(\lambda_K - \theta_{\lambda_K})\right] \\ & + 3g'_{\lambda_1}(\theta)g'_{\lambda_K}(\theta)^2 E\left[(\lambda_1 - \theta_{\lambda_1})(\lambda_K - \theta_{\lambda_K})^2\right], \end{aligned} \tag{37}$$

Accordingly, we give the following theorems that formulate a simple approximation for the central moments of the SCN.

Theorem 1. Let X be the SCN of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$. The mean, the variance and the skewness of X, as AC is satisfied, can be tightly approximated using the mean, the variance and the skewness of the $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ as follows:

$$\mu_X = \frac{\mu_{\lambda_1^{\mathcal{H}_0}}}{\mu_{\lambda_K^{\mathcal{H}_0}}} \tag{38}$$

$$\sigma_X^2 = \frac{\sigma^2_{\lambda_1^{\mathcal{H}_0}}}{\mu^2_{\lambda_K^{\mathcal{H}_0}}} + \frac{\mu^2_{\lambda_1^{\mathcal{H}_0}} \sigma^2_{\lambda_K^{\mathcal{H}_0}}}{\mu^4_{\lambda_K^{\mathcal{H}_0}}} \tag{39}$$

$$\mathcal{S}_X = \frac{1}{\sqrt{\sigma_X^3}} \cdot \left[\frac{\sqrt{\sigma^3_{\lambda_1^{\mathcal{H}_0}}}\mathcal{S}_{\lambda_1^{\mathcal{H}_0}}}{\mu^3_{\lambda_K^{\mathcal{H}_0}}} - \frac{\sqrt{\sigma^3_{\lambda_K^{\mathcal{H}_0}}}\mu^3_{\lambda_1^{\mathcal{H}_0}}\mathcal{S}_{\lambda_K^{\mathcal{H}_0}}}{\mu^6_{\lambda_K^{\mathcal{H}_0}}} \right] \tag{40}$$

Proof. The result follows (35), (36) and (37) while considering $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ asymptotically independent [29]. The mean, the variance and the skewness of $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ are given in Section 3.1. □

Theorem 2. Let X be the SCN of $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$ where $\widehat{\Sigma}_K$ has only one non-unit eigenvalue. The mean, the variance and the skewness of X, as the AC and CC are satisfied, can be tightly approximated using the mean, the variance and the skewness of $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ as follows:

$$\mu_X = \frac{\mu_{\lambda_1^{\mathcal{H}_1}}}{\mu_{\lambda_K^{\mathcal{H}_1}}} \tag{41}$$

$$\sigma_X^2 = \frac{\sigma^2_{\lambda_1^{\mathcal{H}_1}}}{\mu^2_{\lambda_K^{\mathcal{H}_1}}} + \frac{\mu^2_{\lambda_1^{\mathcal{H}_1}} \sigma^2_{\lambda_K^{\mathcal{H}_1}}}{\mu^4_{\lambda_K^{\mathcal{H}_1}}} \tag{42}$$

$$\mathcal{S}_X = -\frac{\sqrt{\sigma^3_{\lambda_K^{\mathcal{H}_1}}}\mu^3_{\lambda_1^{\mathcal{H}_1}}\mathcal{S}_{\lambda_K^{\mathcal{H}_1}}}{\sqrt{\sigma_X^3} \cdot \mu^6_{\lambda_K^{\mathcal{H}_1}}} \tag{43}$$

Proof. The result follows (35), (36) and (37) while considering $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ asymptotically independent [30]. The mean, the variance and the skewness of $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ are given in Section 3.2 □

4.2. Approximating the SCN using GEV

GEV is a flexible 3-parameter distribution used to model the extreme events of a sequence of i.i.d random variables [31]. These parameters are the location (δ), the scale (β) and the shape (ξ). In the following two propositions, we approximate the distribution of the SCN under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses respectively.

Proposition 1. Let X be the SCN of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ with defined skewness $-0.63 \le \mathcal{S}_X < 1.14$ [1]. If AC is satisfied, then the CDF and PDF of X can be asymptotically and tightly approximated respectively by:

$$F(x; \delta, \beta, \xi) = e^{-(1+(\frac{x-\delta}{\beta})\xi)^{-1/\xi}} \tag{44}$$

$$f(x; \delta, \beta, \xi) = \frac{1}{\beta}(1 + (\frac{x-\delta}{\beta})\xi)^{\frac{-1}{\xi}-1} e^{-(1+(\frac{x-\delta}{\beta})\xi)^{-1/\xi}} \tag{45}$$

where ξ, β and δ are defined respectively by:

$$\xi = -0.06393\mathcal{S}_X^2 + 0.3173\mathcal{S}_X - 0.2771 \tag{46}$$

$$\beta = \sqrt{\frac{\sigma_X^2 \xi^2}{g_2 - g_1^2}} \tag{47}$$

$$\delta = \mu_X - \frac{(g_1 - 1)\beta}{\xi} \tag{48}$$

where μ_X, σ_X^2 and $\mathcal{S}_X$ are defined in Theorem 1 and $g_i = \Gamma(1 - i\xi)$.

Proposition 2. Let X be the SCN of $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$ with defined skewness $-0.63 \le \mathcal{S}_X < 1.14$ and $\widehat{\Sigma}_K$ has only one non-unit eigenvalue. If AC and CC are satisfied, then the CDF and PDF of X can be asymptotically and tightly approximated by (44) and (45) respectively. The parameters ξ, β and δ are defined respectively by (46), (47) and (48) with μ_X, σ_X^2 and $\mathcal{S}_X$ are defined in Theorem 2.

It is worth mentioning that the shape, the scale and the location parameters are respectively derived using the skewness, the variance and the mean of the GEV distribution. The moments of the GEV distribution are provided by [32].

5. SCN Detector Analysis

Recall that the SCN is given by:

$$X = \frac{\lambda_1}{\lambda_K}. \tag{49}$$

Denoting by α the decision threshold, then the probability of false alarm (P_{fa}), defined as the probability of detecting the presence of PU while it does not exist, and the detection probability (P_d), defined as the probability of correctly detecting the presence of

[1] Eq. (46) is valid only if $-0.63 \le \mathcal{S}_X < 1.14$ which is true for SCN; however for other skewness intervals the reader may refer to [17]

PU, are, respectively, given by:

$$P_{fa} = P(X \geq \alpha/\mathcal{H}_0), \quad (50)$$
$$P_d = P(X \geq \alpha/\mathcal{H}_1). \quad (51)$$

These probabilities depend on the decision threshold being used. However, if the expressions of the P_{fa} and P_d are previously known, then a threshold could be set according to a required error constraint. If we denote the CDF of X under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses respectively by $F_0(.)$ and $F_1(.)$, then we can write:

$$P_{fa} = 1 - F_0(\alpha), \quad (52)$$
$$P_d = 1 - F_1(\alpha). \quad (53)$$

Consequently, for a given decision threshold ($\hat{\alpha}$) the SCN detector algorithm could be summarized as follows:

Algorithm 1: SCN Detector

Input: $\boldsymbol{Y}$, $\hat{\alpha}$
Output: d
1 compute $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^{\dagger}$;
2 get λ_1 and λ_K of $eig(\boldsymbol{W})$;
3 evaluate $X = \lambda_1/\lambda_K$;
4 decide $d = X \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \hat{\alpha}$;

Performance Probabilities. Based on Propositions 1 and 2, and using (52) and (53), the P_{fa} and P_d are respectively expressed as:

$$P_{fa} = 1 - e^{-(1+(\frac{\alpha-\delta_0}{\beta_0})\xi_0)^{-1/\xi_0}}, \quad (54)$$
$$P_d = 1 - e^{-(1+(\frac{\alpha-\delta_1}{\beta_1})\xi_1)^{-1/\xi_1}}, \quad (55)$$

where δ_0, β_0 and ξ_0 are the location, scale and shape parameters under $\mathcal{H}_0$ and are computed according to proposition 1 while δ_1, β_1 and ξ_1 are the location, scale and shape parameters under $\mathcal{H}_1$ and are computed according to proposition 2.

Decision Threshold. The threshold could be computed using (54) and (55) according to a required error constraint. For example, for a target false alarm probability ($\hat{\gamma}$), the threshold is given by:

$$\hat{\alpha} = \delta_0 + \frac{\beta_0}{\xi_0}\left(-1 + \left[-\ln(1-\hat{\gamma}) \right]^{-\xi_0} \right). \quad (56)$$

6. Numerical validation

In this section, we verify the analytical derivation results through Monte-Carlo simulations. We validate the theoretical analysis presented in sections 3, 4 and 5.

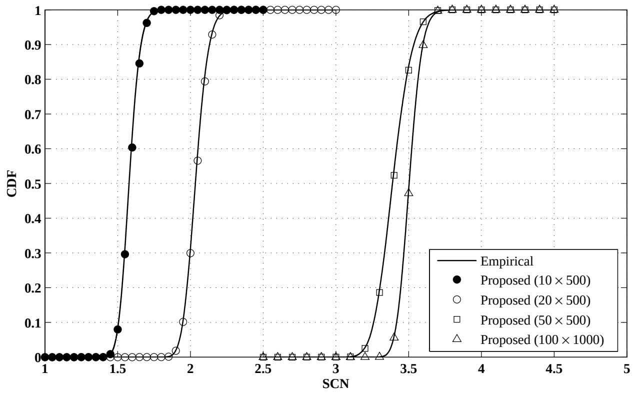


Figure 1. Empirical CDF of the SCN and its corresponding proposed GEV approximation for different values of K and N under $\mathcal{H}_0$ hypothesis.

The simulation results are obtained by generating 10^5 random realizations of $\boldsymbol{Y}$. For $\mathcal{H}_0$ case, the inputs of $\boldsymbol{Y}$ are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2 while for $\mathcal{H}_1$ case the channel is considered flat.

Table 1 shows the accuracy of the analytical approximation of the mean, the variance and the skewness of the SCN provided by Theorems 1 and 2. It can be easily seen that these Theorems provide a good approximation for the statistics of the SCN, however, it could be noticed that the skewness is not perfectly approximated. In fact, the skewness is affected by the slow convergence of the skewness of λ_K that must converge to $-\mathcal{S}_{TW2}$ (i.e. -0.2241) as AC is satisfied. For example, for $K = 50$, the empirical skewness increases from $\mathcal{S}_{\lambda_K} = -0.1504$ to $\mathcal{S}_{\lambda_K} = -0.1819$ as the number of samples increases from $N = 500$ to $N = 1000$. Comparing these results with SCN results in Table 1, one can notice that the empirical and approximated SCN skewness become closer as λ_K skewness converges to that of TW2. Accordingly, Theorems 1 and 2 are good approximations for the mean, the variance and the skewness of the SCN under both hypotheses. It is worth noting that one could approximate the SCN moments using second order bi-variate Taylor series to get a slightly higher accuracy, however, this will cost higher complexity and it is not necessary as shown in Table 1 and the following figures.

Figure 1 shows the empirical CDF of the SCN and its corresponding GEV approximation given by Proposition 1. The results are shown for $K = \{10, 20, 50, 100\}$ antennas and $N = \{500, 1000\}$ samples per antenna. Results show a perfect match between the empirical results and our proposed approximation. Also, it could be noticed that the convergence of the skewness does not affect the approximation and thus the skewness in Theorem 1 holds for this approximation

Table 1. Empirical mean, variance and skewness of the SCN under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses and it corresponding proposed analytical approximation using Theorems 1 and 2 respectively.

	$K \times N$	Empirical			Proposed App.		
		mean	variance	skewness	mean	variance	skewness
$\mathcal{H}_0$	50×500	3.3946	0.0117	0.2639	3.3975	0.0114	0.1652
$\mathcal{H}_1$		12.3363	0.4006	0.1710	12.3139	0.3906	0.0291
$\mathcal{H}_0$	100×500	6.3076	0.0386	0.2992	6.3126	0.0367	0.1740
$\mathcal{H}_1$		34.6345	3.2246	0.1618	34.5387	3.1154	0.0306
$\mathcal{H}_0$	50×1000	2.3386	0.0026	0.2339	2.3396	0.0026	0.1619
$\mathcal{H}_1$		9.6702	0.1205	0.1177	9.6612	0.1184	0.024

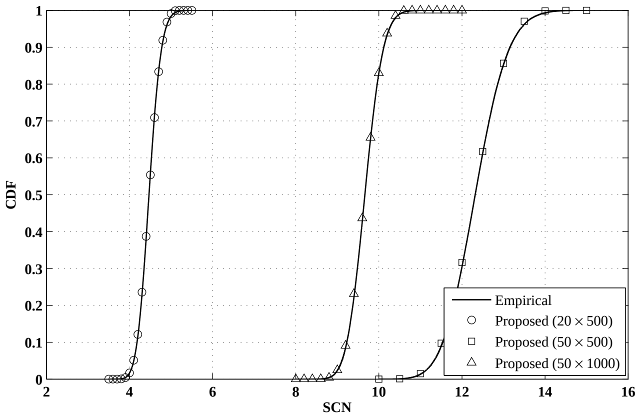


Figure 2. Empirical CDF of the SCN and its corresponding proposed GEV approximation for different values of K and N under $\mathcal{H}_1$ hypothesis with $\rho = -10dB$.

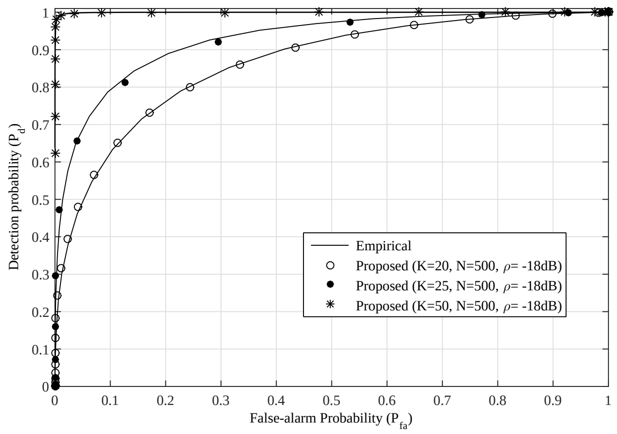


Figure 3. Empirical ROC of the SCN detector and its corresponding proposed approximation for different values of K with $N = 500$ and $\rho = -18dB$.

even though the convergence of the skewness of λ_K is slow.

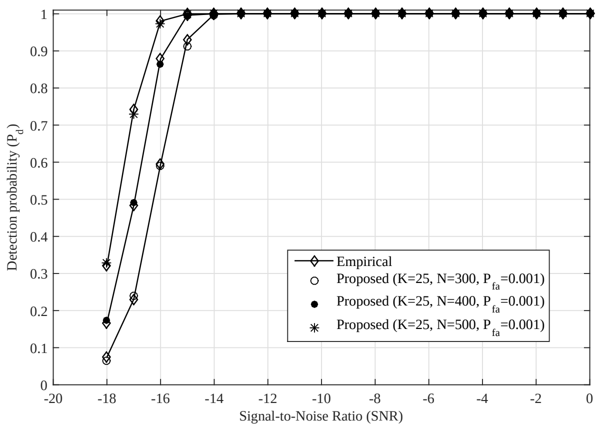
Figure 2 shows the empirical CDF of the SCN and its corresponding GEV approximation given by Proposition 2. The results are shown for $K = \{20, 50\}$ antennas and $N = \{500, 1000\}$ samples per antenna and $\rho = -10dB$. Results show high accuracy in approximating the empirical CDF. Also, the difference in the skewness shown in Table 1 does not affect the approximation.

Figure 3 shows the empirical Receiver Operating Characteristic (ROC) of the SCN detector and its corresponding proposed approximation. The results are shown for $K = \{20, 25, 50\}$ antennas and $N = 500$ samples per antenna for $SNR = -18dB$. Results show that the proposed approximation matches the empirical results with high accuracy. In addition, Fig. 3 shows the gain in the performance as the number of antenna increases.

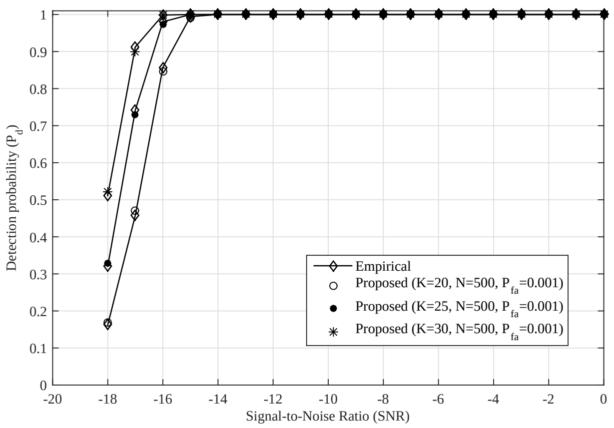
Figure 4 plots the empirical P_d versus SNR (ρ) and its corresponding proposed analytical approximation. P_{fa} is fixed to 0.001 and the threshold is calculated using (56) for different values of $K = \{20, 25, 30\}$ and $N = \{300, 400, 500\}$. Results show the accuracy of the approximation as a function of the SNR for different K and N values. Figure 4(a) shows how the P_d is improved as N increases while Figure 4(b) shows the P_d improvement as K increases. Both Figures show a high P_d when using large number of antennas and relatively large number of samples.

It is worth mentioning that Figures 3 and 4 show high improvement in the system performance by a simple increase of the number of antennas or number of samples used in the sensing process. Accordingly, it is very important to have a simple form for the performance probabilities and thus for the decision threshold so a CR system with large number of antennas[2] can dynamically adapt its threshold

[2]e.g. Massive MIMO, Large-scale distributed antenna systems, Cooperation between large number of SU.



(a) P_d versus ρ for $K = 25$ and N varies



(b) P_d versus ρ for $N = 500$ and K varies

Figure 4. Empirical P_d of the SCN detector as a function of SNR and its corresponding proposed approximation for different values of N and K with $P_{fa} = 0.001$.

according to pre-defined error constraints and channel conditions.

7. Conclusion

In this paper, we have considered the SCN detector for large number of antennas in cognitive radios. We have derived the asymptotic mean, variance and skewness of the SCN using those of the extreme eigenvalues of the sample covariance matrix by means of bi-variate Taylor expansion. A simple closed form approximation for the distribution of the SCN under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses are proposed. This approximation is based on the extreme value theory distributions and uses results from random matrix theory. Consequently, simple forms for the false-alarm probability, detection probability and the decision threshold are derived for real-time computations such that a CR system with large number of antennas can dynamically adapt its threshold according to pre-defined error constraints and channel conditions. In addition to their simple forms, simulation results show high accuracy of the proposed approximation for different number of antennas and different number of samples on various SNR values.

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