1. Introduction

Autonomic computing (AC) imitates and simulates the natural intelligence possessed by the human autonomic nervous system using generic computers. This indicates that the nature of software in AC is the simulation and embodiment of human behaviors, and the extension of human capability, reachability, persistency, memory, and information processing speed. AC was first proposed by IBM in 2001 where it is defined as

"Autonomic computing is an approach to self-managed computing systems with a minimum of human interference. The term derives from the body’s autonomic nervous system, which controls key functions without conscious awareness or involvement" [1].

AC in our recent investigations [2–8] is generally described as self-*. Formally, let self-* be the set of self-__’s. Each self-__ to be an element in self-* is called a self-* facet. That is,

\[
\text{self-*} = \{\text{self-__} \mid \text{self-__ is a self-* facet}\} \quad (1)
\]

We see that self-CHOP is composed of four self-* facets of self-configuration, self-healing, self-optimization and self-protection. Hence, self-CHOP is a subset of self-*. That is, self-CHOP = \{self-configuration, self-healing, self-optimization, self-protection\} \subseteq self-*. Every self-* facet must satisfy some certain criteria, so-called self-* properties.

In its AC manifesto, IBM proposed eight facets setting forth an AS known as self-awareness, self-configuration, self-optimization, self-maintenance, self-protection (security and integrity), self-adaptation, self-resource-allocation and open-standard-based [1]. In other words, consciousness (self-awareness) and non-imperative (goal-driven) behaviors are the main features of autonomic systems (ASs).

In this paper we will specify ASs and self-* and then move on to consider products and coproducts of ASs. All of this material is taken as an investigation of our category, the category of ASs, which we call AS.

2. Outline

In the paper, we attempt to make the presentation as self-contained as possible, although familiarity with the notion of self-* in ASs is assumed. Acquaintance with the associated notion of algebraic language is useful for recognizing the results, but is almost everywhere not strictly necessary.

The rest of this paper is organized as follows: Section 3 presents the notion of autonomic systems (ASs). In section 4, self-* actions in ASs are specified. In section 5, products and coproducts of ASs are considered. Some universal properties of ASs are investigated in section 6. Finally, a short summary is given in section 7.
3. Autonomic Systems (ASs)

We can think of an AS as a collection of states \( x \in \text{AS} \), each of which is recognizable as being in AS and such that for each pair of named states \( x, y \in \text{AS} \) we can tell if \( x = y \) or not. The symbol \( \subseteq \) denotes the AS with no states.

If \( \text{AS}_1 \) and \( \text{AS}_2 \) are ASs, we say that \( \text{AS}_1 \) is a sub-system of \( \text{AS}_2 \), and write \( \text{AS}_1 \subseteq \text{AS}_2 \), if every state of \( \text{AS}_1 \) is a state of \( \text{AS}_2 \). Checking the definition, we see that for any system AS, we have sub-systems \( \emptyset \subseteq \text{AS} \) and \( \text{AS} \subseteq \text{AS} \).

We can use system-builder notation to denote sub-systems. For example the autonomic system can be written \( \{x \in \text{AS} \mid x \text{ is a state of AS}\} \).

The symbol \( \exists \) means “there exists”. So we can write the autonomic system as \( \{x \in \text{AS} \mid \exists y \text{ is a final state such that } \text{self-}^*\text{action}(x) = y\} \).

The symbol \( \forall \) means “exists a unique”. So the statement “\( \forall x \in \text{AS} \) is an initial state” means that there is one and only one state to be a start one, that is, the state of the autonomic system before any self-* action is processed.

Finally, the symbol \( \forall \) means “for all”. So the statement “\( \forall x \in \text{AS} \) \( \exists y \in \text{AS} \) such that \( \text{self-}^*\text{action}(x) = y \)” means that for every state of autonomic system there is the next one.

In the paper, we use the \( \text{def} \) notation “\( \text{AS}_1 \text{def} \text{AS}_2 \)” to mean something like “define AS_1 to be AS_2”. That is, a \( \text{def} \) declaration is not denoting a fact of nature (like \( 1 + 2 = 3 \)), but our formal notation. It just so happens that the notation above, such as Self-CHOP \( \text{def} \{\text{self-configuration, self-healing, self-optimization, self-protection}\} \), is a widely chosen.

4. Self-* Actions of Autonomic Systems

If \( \text{AS} \) and \( \text{AS}' \) are sets of autonomic system states, then a self-* action \( \text{self-}^*\text{action} \) from \( \text{AS} \) to \( \text{AS}' \), denoted self-*action: \( \text{AS} \rightarrow \text{AS}' \), is a mapping that sends each state \( x \) in \( \text{AS} \) to a state of \( \text{AS}' \), denoted self-*action\( (x) \in \text{AS}' \). We call \( \text{AS} \) the domain of self-*action and we call \( \text{AS}' \) the codomain of self-*action.

Note that the symbol \( \text{AS}' \), read “AS-prime”, has nothing to do with calculus or derivatives. It is simply notation that we use to name a symbol that is suggested as being somehow like \( \text{AS} \). This suggestion of consanguinity between \( \text{AS} \) and \( \text{AS}' \) is meant only as an aid for human cognition, and not as part of the mathematics. For every state \( x \) in \( \text{AS} \), there is exactly one arrow emanating from \( x \), but for a state \( y \) in \( \text{AS}' \), there can be several arrows pointing to \( y \), or there can be no arrows pointing to \( y \).

Suppose that \( \text{AS}' \subseteq \text{AS} \) is a sub-system. Then we can consider the self-* action \( \text{AS} \rightarrow \text{AS}' \) given by sending every state of \( \text{AS}' \) to “itself” as a state of \( \text{AS} \). For example if \( \text{AS} = \{a,b,c,d,e,f\} \) and \( \text{AS}' = \{b,d,e\} \) then \( \text{AS}' \subseteq \text{AS} \) and we turn that into the self-* action \( \text{AS} \rightarrow \text{AS}' \) given by \( b \mapsto b, d \mapsto d, e \mapsto e \). This kind of arrow, \( \mapsto \), is read aloud as “maps to”. A self-* action \( \text{self-}^*\text{action} : \text{AS} \rightarrow \text{AS}' \) means a rule for assigning to each state \( x \in \text{AS} \) a state self-*action\( (x) \in \text{AS}' \). We say that “\( x \) maps to self-*action\( (x) \)” and write \( x \mapsto \text{self-}^*\text{action}(x) \).

As a matter of notation, we can sometimes say something like the following: Let \( \text{self-}^*\text{action}: \text{AS} ' \subseteq \text{AS} \) be a sub-system. Here we are making clear that \( \text{AS}' \) is a sub-system of \( \text{AS} \), but that \( \text{self-}^*\text{action} \) is the name of the associated self-* action.

Given a self-* action \( \text{self-}^*\text{action} : \text{AS} \rightarrow \text{AS}' \), the states of \( \text{AS}' \) that have at least one arrow pointing to them are said to be in the image of self-*action; that is we have \( \text{im}(\text{self-}^*\text{action}) \) \( \text{def} \) \( \{y \in \text{AS}' \mid \exists x \in \text{AS} \text{ such that } \text{self-}^*\text{action}(x) = y\} \). Given self-* action \( \text{AS} \rightarrow \text{AS}' \) and \( \text{self-}^*\text{action}' : \text{AS}' \rightarrow \text{AS}'' \), where the codomain of self-*action is the same set of autonomic system states as the domain of self-*action' (namely \( \text{AS}' \)), we say that self-*action and self-*action' are composable

\[
\text{AS} \text{self-}^*\text{action} \rightarrow \text{AS}' \text{self-}^*\text{action}' \rightarrow \text{AS}''
\]

The composition of self-*action and self-*action' is denoted by self-*action' \( \circ \) self-*action : \( \text{AS} \rightarrow \text{AS}'' \).

We write \( \text{Hom}_{\text{AS}}(\text{AS}, \text{AS}') \) to denote the set of self-*actions \( \text{AS} \rightarrow \text{AS}' \). Two self-* actions \( \text{self-}^*\text{action}, \text{self-}^*\text{action}' : \text{AS} \rightarrow \text{AS}' \) are equal if and only if for every state \( x \in \text{AS} \) we have \( \text{self-}^*\text{action}(x) = \text{self-}^*\text{action}'(x) \).

We define the identity self-*action on \( \text{AS} \), denoted \( \text{id}_\text{AS} : \text{AS} \rightarrow \text{AS} \), to be the self-* action such that for all \( x \in \text{AS} \) we have \( \text{id}_\text{AS}(x) = x \).

A self-*action \( \text{AS} \rightarrow \text{AS}' \) is called an isomorphism, denoted self-*action : \( \text{AS} \overset{\cong}{\rightarrow} \text{AS}' \), if there exists a self-* action self-*action' : \( \text{AS}' \rightarrow \text{AS} \) such that self-*action' \( \circ \) self-*action = idAS and self-*action \( \circ \) self-*action' = idAS. We also say that self-*action is invertible and we say that self-*action' is the inverse of self-*action. If there exists an isomorphism \( \text{AS} \overset{\cong}{\rightarrow} \text{AS}' \) we say that \( \text{AS} \) and \( \text{AS}' \) are isomorphic autonomic systems and may write \( \text{AS} \cong \text{AS}' \).

**Proposition 1.** The following facts hold about isomorphism.

1. Any autonomic system \( \text{AS} \) is isomorphic to itself; i.e. there exists an isomorphism \( \text{AS} \overset{\cong}{\rightarrow} \text{AS} \).

2. For any autonomic systems \( \text{AS} \) and \( \text{AS}' \), if \( \text{AS} \) is isomorphic to \( \text{AS}' \) then \( \text{AS}' \) is isomorphic to \( \text{AS} \).

3. For any autonomic systems \( \text{AS}, \text{AS}' \) and \( \text{AS}'' \), if \( \text{AS} \) is isomorphic to \( \text{AS}' \) and \( \text{AS}' \) is isomorphic to \( \text{AS}'' \) then \( \text{AS} \) is isomorphic to \( \text{AS}'' \).

**Proof:**

1. The identity self-* action \( \text{id}_\text{AS} : \text{AS} \rightarrow \text{AS} \) is invertible; its inverse is \( \text{id}_\text{AS} \) because \( \text{id}_\text{AS} \circ \text{id}_\text{AS} = \text{id}_\text{AS} \).

2. If self-*action : \( \text{AS} \rightarrow \text{AS}' \) is invertible with inverse self-*action' : \( \text{AS}' \rightarrow \text{AS} \) then self-*action' is an isomorphism with inverse self-*action.

3. If self-*action : \( \text{AS} \rightarrow \text{AS}' \) and self-*action : \( \text{AS}' \rightarrow \text{AS}'' \) are each invertible with inverses self-*action' : \( \text{AS}' \rightarrow \text{AS} \) and self-*action' : \( \text{AS}'' \rightarrow \text{AS} \) then the following calculations show that self-*action \( \circ \) self-*action is invertible with inverse self-*action' \( \circ \) self-*action'.
are isomorphic if and only if

Suppose that

Then we

So, in particular

is an isomorphism. We can prove

We say this is a diagram of autonomic systems if each

and

is an autonomic system and each of

is a self-* action. We say this diagram commutes if

In this case we refer to it as a commutative square of autonomic systems.

5. Products and Coproducts of Autonomic Systems

Let \( AS \) and \( AS' \) be autonomic systems. The product of \( AS \) and \( AS' \), denoted \( AS \times AS' \), is defined as the autonomic system of ordered pairs \( (x, y) \) where states of \( x \in AS \) and \( y \in AS' \). Symbolically, \( AS \times AS' = \{(x, y) | x \in AS, y \in AS'\} \). There are two natural projection actions of self-* to be self-* action \( \times \) to be autonomic system \( AS \times AS' \rightarrow AS \) and self-* action \( \times \) to be autonomic system \( AS \times AS' \rightarrow AS' \)

For illustration, suppose that \( \{a, b, c\} \) are states in \( AS \) and \( \{d, e\} \) in \( AS' \), the states are happening in such autonomic systems. Thus, \( AS \) and \( AS' \), which are running concurrently, can be specified by \( AS/AS' \overset{\text{def}}{=} \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\} \). Note that the symbol \( "|" \) is used to denote concurrency of states existing at the same time. We define self-* actions as disable\((d, e)\) and disable\((a, b, c)\) to be able to drop out relevant states.

\[
\{\{a, b, c\}, \{d, e\}\} 
\]

\[
\{d, e\} 
\]

5. Products and Coproducts of Autonomic Systems

Let \( AS \) and \( AS' \) be autonomic systems. The product of \( AS \) and \( AS' \), denoted \( AS \times AS' \), is defined as the autonomic system of ordered pairs \( (x, y) \) where states of \( x \in AS \) and \( y \in AS' \). Symbolically, \( AS \times AS' = \{(x, y) | x \in AS, y \in AS'\} \). There are two natural projection actions of self-* to be self-* action \( \times \) to be autonomic system \( AS \times AS' \rightarrow AS \) and self-* action \( \times \) to be autonomic system \( AS \times AS' \rightarrow AS' \)

For illustration, suppose that \( \{a, b, c\} \) are states in \( AS \) and \( \{d, e\} \) in \( AS' \), the states are happening in such autonomic systems. Thus, \( AS \) and \( AS' \), which are running concurrently, can be specified by \( AS/AS' \overset{\text{def}}{=} \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\} \). Note that the symbol \( "|" \) is used to denote concurrency of states existing at the same time. We define self-* actions as disable\((d, e)\) and disable\((a, b, c)\) to be able to drop out relevant states.

\[
\{\{a, b, c\}, \{d, e\}\} 
\]

\[
\{d, e\} 
\]
Proposition 3. Let $AS$ and $AS'$ be autonomic systems. For any autonomic system $AS''$ and actions self-\textit{action}$_1 : AS'' \rightarrow AS$ and self-\textit{action}$_2 : AS'' \rightarrow AS'$, there exists a unique action $AS'' \rightarrow AS \times AS'$ such that the following diagram commutes

\[
\begin{array}{c}
\text{self-\textit{action}$_1$} \\
\downarrow \quad \downarrow \quad \downarrow \\
AS \\
\text{self-\textit{action}$_2$} \\
\downarrow \\
AS'
\end{array}
\]

\[
\vdash \text{self-\textit{action}$_3$} \\
\downarrow \\
AS'' \\
\text{self-\textit{action}$_4$} \\
\downarrow \\
AS'' \rightarrow AS \times AS'
\]

We might write the unique action as

\[
(\text{self-\textit{action}$_3$}, \text{self-\textit{action}$_4$}) : AS'' \rightarrow AS \times AS'
\]

\textbf{Proof:} Suppose given self-\textit{action}$_3$ and self-\textit{action}$_4$ as above. To provide an action $z : AS'' \rightarrow AS \times AS'$ is equivalent to providing a state $z(a) \in AS \times AS'$ for each $a \in AS''$. We need such an action for which self-\textit{action}$_1 \circ z = \text{self-\textit{action}$_3$}$ and self-\textit{action}$_2 \circ z = \text{self-\textit{action}$_4$}$. A state of $AS \times AS'$ is an ordered pair $(x, y)$, and we can use $z(a) = (x, y)$ if and only if $x = \text{self-\textit{action}$_1$}(a)$ and $y = \text{self-\textit{action}$_2$}(a)$. So it is necessary and sufficient to define

\[
(\text{self-\textit{action}$_3$}, \text{self-\textit{action}$_4$}) = ((a) \mapsto (\text{self-\textit{action}$_3$}(a), \text{self-\textit{action}$_4$}(a))) \quad \forall a \in AS''.
\]

Q.E.D.

Given autonomic systems $AS$, $AS'$, and $AS''$, and actions self-\textit{action}$_1 : AS'' \rightarrow AS$ and self-\textit{action}$_4 : AS'' \rightarrow AS'$, there is a unique action $AS'' \rightarrow AS \times AS'$ that commutes with self-\textit{action}$_1$ and self-\textit{action}$_4$. We call it the \textit{induced action} $AS'' \rightarrow AS \times AS'$, meaning the one that arises in light of self-\textit{action}$_1$ and self-\textit{action}$_4$.

For example, as mentioned above autonomic systems $AS = \{a, b, c\}$, $AS' = \{d, e\}$ and $AS | AS' \overset{\text{def}}{=} \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$. For an autonomic system $AS'' = \emptyset$, which stops running, we define self-\textit{action}$_1$ and self-\textit{action}$_2$ to enable $(a, d)$ and $(b, e)$ to be able to add further relevant states. Then there exists a unique action

\[
(\text{enable}((a, d), (a, e), (b, d), (b, e), (c, d), (c, e)))
\]

such that the following diagram commutes

\[
\begin{array}{c}
\{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{a, b, c\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{d, e\}
\end{array}
\]

We might write the unique action as

\[
[\text{self-\textit{action}$_3$}, \text{self-\textit{action}$_4$}] : AS | AS' \rightarrow AS''
\]

\textbf{Proposition 4.} Let $AS$ and $AS'$ be autonomic systems. For any autonomic system $AS''$ and actions self-\textit{action}$_1 : AS \rightarrow AS''$ and self-\textit{action}$_4 : AS' \rightarrow AS''$, there exists a unique action $AS \cup AS' \rightarrow AS''$ such that the following diagram commutes

\[
\begin{array}{c}
\text{self-\textit{action}$_1$} \\
\downarrow \\
AS \cup AS'
\end{array}
\]

\[
\begin{array}{c}
\text{self-\textit{action}$_2$} \\
\downarrow \\
AS''
\end{array}
\]

\[
\vdash \text{self-\textit{action}$_3$} \\
\downarrow \\
AS'' \rightarrow AS \times AS'
\]

\[
\text{self-\textit{action}$_4$} \\
\downarrow \\
AS'' \rightarrow AS \times AS'
\]

\textbf{Proof:} Suppose given self-\textit{action}$_1$, self-\textit{action}$_4$ as above. To provide an action $z : AS \cup AS' \rightarrow AS''$ is equivalent to providing a state self-\textit{action}$_3(m) \in AS''$ is for each $m \in AS \cup AS'$. We need such an action such that $z \circ \text{self-\textit{action}$_1$} = \text{self-\textit{action}$_3$}$ and $z \circ \text{self-\textit{action}$_4$} = \text{self-\textit{action}$_4$}$. But each state $m \in AS \cup AS'$ is either of the form self-\textit{action}$_1x$ or self-\textit{action}$_2y$, and cannot be of both forms. So we assign

\[
[\text{self-\textit{action}$_3$}, \text{self-\textit{action}$_4$}] = \begin{cases} 
\text{self-\textit{action}$_1x$} & \text{if } m = \text{self-\textit{action}$_1x$} \\
\text{self-\textit{action}$_2y$} & \text{if } m = \text{self-\textit{action}$_2y$}
\end{cases}
\]

This assignment is necessary and sufficient to make all relevant diagrams commute.

Q.E.D.

For example, as mentioned above autonomic systems $AS = \{a, b, c\}$, $AS' = \{d, e\}$ and $AS | AS' \overset{\text{def}}{=} \{a, b, c, d, e\}$. For an autonomic system $AS'' = \emptyset$, which stops running, we define self-\textit{action}$_1$ as disable$(d,e)$ and disable$(a,b,c)$ to drop out relevant states. Then there exists a unique

union” of $AS$ and $AS'$, i.e. the autonomic system for which a state is either a state of $AS$ or a state of $AS'$. If something is a state of both $AS$ and $AS'$ then we include both copies, and distinguish between them, in $AS \cup AS'$. There are two natural inclusion actions self-\textit{action}$_1 : AS \rightarrow AS \cup AS'$ and self-\textit{action}$_2 : AS' \rightarrow AS \cup AS'$.
action $\text{disable}(a, b, c, d, e)$ such that the following diagram commutes

$$
\begin{array}{c}
\text{disable}(a, b, c) \\
\text{\{a, b, c\}} \\
\text{\{a, b, c, d, e\}} \\
\text{enable}(d, e) \\
\end{array}
\begin{array}{c}
\text{\{d, e\}} \\
\end{array}
\begin{array}{c}
\text{\{a, b, c, d, e\}} \\
\text{enable}(a, b, c) \\
\end{array}
$$

(11)

6. Universal Properties

We denote the coproduct of two autonomic systems $AS$ and $AS'$ by the notation $AS + AS'$ rather than $AS \sqcup AS'$. It is a reasonable notation in general, and one that is often used.

The following isomorphisms exist for any autonomic systems $AS$, $AS'$, and $AS''$:

$$
\begin{align*}
&AS + 0 \cong AS \\
&AS + AS' \cong AS' + AS \\
&(AS + AS') + AS'' \cong AS + (AS' + AS'') \\
&AS \times 0 \cong 0 \\
&AS \times 1 \cong AS \\
&(AS \times AS') \times AS'' \cong AS \times (AS' \times AS'') \\
&AS \times (AS' + AS'') \cong (AS \times AS') + (AS \times AS'') \\
&AS_0 \cong 1 \\
&AS^* \cong AS \\
&0^* \cong 0 \\
&1^* \cong 1 \\
&AS^{AS'} \cong AS^{AS'} \times AS^{AS''} \\
&(AS^{AS'})^{AS''} \cong AS^{AS'} \times AS^{AS''}
\end{align*}
$$

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References


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